Mathematics 325 Exam II Name: Dr. Grow

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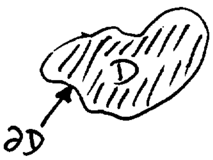
1.(25 pts.) (a) State and prove the weak maximum principle for solutions to Laplace's equation.

(b) Show (by exhibiting an appropriate example) that solutions to the wave equation need not obey a maximum principle.

(a)



If $u = u(x,y)$ is a solution to $u_{xx} + u_{yy} = 0$ in an open bounded set D of the plane, and u is continuous on $\bar{D} = D \cup \partial D$ then $\max\limits_{(x,y)\in\bar{D}} u(x,y) = \max\limits_{(x,y)\in\partial D} u(x,y)$.

Proof: Let $\varepsilon > 0$ and consider $v(x,y) = u(x,y) + \varepsilon(x^2+y^2)$. Suppose v has a maximum value at an (interior) point (x_0,y_0) of D. Then $v_{xx}(x_0,y_0) \le 0$ and $v_{yy}(x_0,y_0) \le 0$ so

$$0 \ge v_{xx}(x_0,y_0) + v_{yy}(x_0,y_0) = u_{xx}(x_0,y_0) + u_{yy}(x_0,y_0) + 4\varepsilon.$$

But then $0 > -4\varepsilon \ge u_{xx}(x_0,y_0) + u_{yy}(x_0,y_0)$ which contradicts the fact that u solves Laplace's equation in D. Thus the maximum value of v on $\bar{D}$ occurs at a point of ∂D. Hence

$$\max_{(x,y)\in\bar{D}} u(x,y) \le \max_{(x,y)\in\bar{D}} \Big[\overbrace{u(x,y) + \varepsilon(x^2+y^2)}^{v(x,y)}\Big] = \max_{(x,y)\in\partial D}\left[u(x,y) + \varepsilon(x^2+y^2)\right]$$

which implies $\max\limits_{(x,y)\in\bar{D}} u(x,y) \le \varepsilon M + \max\limits_{(x,y)\in\partial D} u(x,y)$ where

$M = \max\limits_{(x,y)\in\partial D}(x^2+y^2) < \infty$ since D is bounded. Since $\varepsilon > 0$ can be made arbitrarily small, it follows that $\max\limits_{(x,y)\in\bar{D}} u(x,y) \le \max\limits_{(x,y)\in\partial D} u(x,y)$.

But $\partial D \subseteq \bar{D}$ so the reverse inequality $\max\limits_{(x,y)\in\partial D} u(x,y) \le \max\limits_{(x,y)\in\bar{D}} u(x,y)$

(OVER)

is clear. Thus, the desired conclusion follows:

$$\max_{(x,y)\in\bar{D}} u(x,y) = \max_{(x,y)\in\partial D} u(x,y).$$

(b) Consider $u(x,t) = \sin(\pi x)\sin(\pi c t)$ on the rectangle $\bar{R}: 0 \le x \le 1,\ 0 \le t \le 1/c$ in the xt-plane. Then

$$u_{tt} - c^2 u_{xx} = -c^2\pi^2 \sin(\pi c t)\sin(\pi x) - c^2\left[-\pi^2 \sin(\pi x)\sin(\pi c t)\right] = 0$$

so u solves the wave equation in $\bar{R}$. However

$$\max_{(x,t)\in\partial R} u(x,t) = 0 < 1 = \max_{(x,t)\in\bar{R}} u(x,t).$$

2. (25 pts.) Solve the diffusion equation in the upper halfplane subject to the initial condition

$$\phi(x) = \begin{cases} 1 & \text{if } |x| < 2, \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

Write your answer in terms of

$$\mathrm{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-p^2}\, dp.$$

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)\, dy \quad \text{for } -\infty < x < \infty \text{ and } 0 < t < \infty.$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-2}^{2} e^{-\frac{(x-y)^2}{4kt}}\, dy \quad \left\{ \begin{array}{l} \text{Let } p = \frac{x-y}{\sqrt{4kt}}. \text{ Then } dp = \frac{-dy}{\sqrt{4kt}}. \\ y = +2 \Rightarrow p = \frac{x-2}{\sqrt{4kt}}. \\ y = -2 \Rightarrow p = \frac{x+2}{\sqrt{4kt}}. \end{array} \right.$$

$$\therefore u(x,t) = \frac{1}{\sqrt{\pi}} \int_{\frac{x+2}{\sqrt{4kt}}}^{\frac{x-2}{\sqrt{4kt}}} e^{-p^2} (-dp) = \frac{1}{\sqrt{\pi}} \int_{\frac{x-2}{\sqrt{4kt}}}^{\frac{x+2}{\sqrt{4kt}}} e^{-p^2}\, dp$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\frac{x+2}{\sqrt{4kt}}} e^{-p^2}\, dp - \frac{1}{\sqrt{\pi}} \int_0^{\frac{x-2}{\sqrt{4kt}}} e^{-p^2}\, dp$$

$$= \boxed{\frac{1}{2} \mathrm{Erf}\left(\frac{x+2}{\sqrt{4kt}}\right) - \frac{1}{2} \mathrm{Erf}\left(\frac{x-2}{\sqrt{4kt}}\right)}.$$

3. (25 pts.) Consider a thin metal rod of length 1, insulated along its sides but not at its ends, which initially is at temperature 25. Suddenly both ends are plunged into a bath of temperature 0.

(a) Write the partial differential equation, boundary conditions, and initial condition that govern the temperature of the rod.

(b) Find a formula for the temperature $u(x,t)$ of the rod at position x in $[0,1]$ and at time $t \geq 0$. You may assume that

$$(\dagger) \quad 25 = \frac{100}{\pi}\left[\sin(\pi x) + \frac{1}{3}\sin(3\pi x) + \frac{1}{5}\sin(5\pi x) + \dots\right] \quad \text{for } 0 < x < 1.$$

(a) ① $u_t - k u_{xx} = 0$ in $0<x<1$, $0<t<\infty$,

②-③ $u(0,t) = 0 = u(1,t)$ for $t \geq 0$,

④ $u(x,0) = 25$ for $0<x<1$.

(b) $u(x,t) = X(x)T(t)$ leads to $\begin{cases} X''(x) + \lambda X(x) = 0, & \overbrace{X(0) = 0 = X(1)}^{\text{Dirichlet B.C.'s.}} \\ T'(t) + k\lambda T(t) = 0 \end{cases}$

Eigenvalues: $\lambda_n = (n\pi)^2$
Eigenfunctions: $X_n(x) = \sin(n\pi x)$ $\Big\}$ $n = 1, 2, 3, \dots$

Therefore $T_n(t) = e^{-k(n\pi)^2 t}$ so

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-kn^2\pi^2 t} \quad \text{solves ①-②-③.}$$

$$25 \overset{\text{Want}}{=} u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \quad \text{for all } 0<x<1.$$

By inspection (and comparing with identity ($\dagger$) above) we see that $b_n = \begin{cases} 0 & \text{if } n = 2m, \\ \dfrac{100}{\pi(2m+1)} & \text{if } n = 2m+1. \end{cases}$

Thus $\boxed{u(x,t) = \dfrac{100}{\pi} \displaystyle\sum_{m=0}^{\infty} \dfrac{\sin((2m+1)\pi x) e^{-k(2m+1)^2\pi^2 t}}{2m+1}}$.

4.(25 pts.) Show that any solution in the xt-plane of the damped wave equation

$$\rho u_{tt} - Tu_{xx} + ru_t = 0$$

(where ρ, T, and r are positive constants) has an energy function

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty}[\rho u_t^2(x,t) + Tu_x^2(x,t)]dx$$

that is nonincreasing.

$$\frac{dE}{dt} = \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}\left[\rho u_t^2(x,t) + Tu_x^2(x,t)\right]dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left[\rho u_t^2(x,t) + Tu_x^2(x,t)\right]dx$$

$$= \int_{-\infty}^{\infty}\left[\rho u_t(x,t)u_{tt}(x,t) + Tu_x(x,t)u_{xt}(x,t)\right]dx$$

$$= \int_{-\infty}^{\infty}\left[u_t(x,t)\left\{Tu_{xx}(x,t) - ru_t(x,t)\right\} + Tu_x(x,t)u_{xt}(x,t)\right]dx$$

$$= -r\int_{-\infty}^{\infty}u_t^2(x,t)dx + T\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left(u_t(x,t)u_x(x,t)\right)dx$$

$$= -r\int_{-\infty}^{\infty}u_t^2(x,t)dx + T\left[\lim_{x\to\infty}u_t(x,t)u_x(x,t) - \lim_{x\to-\infty}u_t(x,t)u_x(x,t)\right].$$

Assuming that $\lim_{|x|\to\infty} u_t(x,t)u_x(x,t) = 0$ for all real t, we have

$$\frac{dE}{dt} = -r\int_{-\infty}^{\infty}u_t^2(x,t)dx \le 0 .$$

Thus E is a nonincreasing function of t.