1. (25 pts.) (a) State and prove the weak maximum principle for solutions to Laplace's equation.

(b) Show (by exhibiting an appropriate example) that solutions to the wave equation need not obey a maximum principle.

(a) If \( u = u(x,y) \) is a solution to \( u_{xx} + u_{yy} = 0 \) in an open bounded set \( D \) of the plane, and \( u \) is continuous on \( \overline{D} = D \cup \partial D \)

then \( \max_{(x,y)\in\overline{D}} u(x,y) = \max_{(x,y)\in D} u(x,y) \).

Proof: Let \( \varepsilon > 0 \) and consider \( v(x,y) = u(x,y) + \varepsilon (x^2 + y^2) \). Suppose \( v \) has a maximum value at an (interior) point \( (x_0, y_0) \) of \( D \). Then \( v_{xx}(x_0, y_0) \leq 0 \)

and \( v_{yy}(x_0, y_0) \leq 0 \) so

\[
0 \geq v_{xx}(x_0, y_0) + v_{yy}(x_0, y_0) = u_{xx}(x_0, y_0) + u_{yy}(x_0, y_0) + 4\varepsilon .
\]

But then \( 0 > -4\varepsilon \geq u_{xx}(x_0, y_0) + u_{yy}(x_0, y_0) \) which contradicts the fact that \( u \) solves Laplace's equation in \( D \). Thus the maximum value of \( v \) on \( \overline{D} \) occurs at a point of \( \partial D \). Hence

\[
\max_{(x,y)\in\overline{D}} u(x,y) = \max_{(x,y)\in \overline{D}} \left[ u(x,y) + \varepsilon (x^2 + y^2) \right] = \max_{(x,y)\in \overline{D}} \left[ u(x,y) + \varepsilon (x^2 + y^2) \right] = \max_{(x,y)\in D} u(x,y) .
\]

which implies \( \max_{(x,y)\in\overline{D}} u(x,y) \leq \varepsilon M + \max_{(x,y)\in D} u(x,y) \) where

\[
M = \max_{(x,y)\in \partial D} (x^2 + y^2) < \infty \text{ since } D \text{ is bounded}. \text{ Since } \varepsilon > 0 \text{ can be made arbitrarily small, it follows that } \max_{(x,y)\in \overline{D}} u(x,y) \leq \max_{(x,y)\in \partial D} u(x,y) .
\]

But \( \partial D \subseteq \overline{D} \) so the reverse inequality \( \max_{(x,y)\in \overline{D}} u(x,y) \leq \max_{(x,y)\in D} u(x,y) \) holds.

(Over)
is clear. Thus, the desired conclusion follows:

$$\max_{(x,y) \in \overline{D}} u(x,y) = \max_{(x,y) \in \partial D} u(x,y).$$

(b) Consider $u(x,t) = \sin(\pi x) \sin(\pi ct)$ on the rectangle $\overline{R} : 0 \leq x \leq 1, 0 \leq t \leq 1/c$ in the $xt$-plane. Then

$$u_{tt} - c^2 u_{xx} = -\pi^2 \sin(\pi x) \sin(\pi ct) \sin(\pi x) = c^2 \left[ -\pi^2 \sin(\pi x) \sin(\pi ct) \right] = 0$$

so $u$ solves the wave equation in $\overline{R}$. However

$$\max_{(x,t) \in \overline{R}} u(x,t) = 0 < 1 = \max_{(x,t) \in \overline{R}} u(x,t).$$
2. (25 pts.) Solve the diffusion equation in the upper halfplane subject to the initial condition

\[
\phi(x) = \begin{cases} 
1 & \text{if } |x| < 2, \\
0 & \text{if } |x| \geq 2.
\end{cases}
\]

Write your answer in terms of

\[
\text{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-p^2} \, dp.
\]

\[
u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y) \, dy
\]

for \(-\infty < x < \infty\) and \(0 < t < \infty\).

\[
\Rightarrow u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-2}^{2} e^{-\frac{(x-y)^2}{4kt}} \, dy
\]

\[
\text{Let } p = \frac{x-y}{\sqrt{4kt}}. \text{ Then } \, dp = \frac{1}{\sqrt{4kt}} \, dy.
\]

\[
y = +2 \Rightarrow p = \frac{x-2}{\sqrt{4kt}}
\]

\[
y = -2 \Rightarrow p = \frac{x+2}{\sqrt{4kt}}
\]

\[
\therefore u(x,t) = \frac{1}{\sqrt{\pi}} \int_{\frac{x-2}{\sqrt{4kt}}}^{\frac{x+2}{\sqrt{4kt}}} e^{-p^2} (-dp) = \frac{1}{\sqrt{\pi}} \int_{\frac{x+2}{\sqrt{4kt}}}^{\frac{x+2}{\sqrt{4kt}}} e^{-p^2} \, dp - \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x-2}{\sqrt{4kt}}} e^{-p^2} \, dp
\]

\[
= \frac{1}{2} \text{Erf} \left( \frac{x+2}{\sqrt{4kt}} \right) - \frac{1}{2} \text{Erf} \left( \frac{x-2}{\sqrt{4kt}} \right).
\]
3. (25 pts.) Consider a thin metal rod of length 1, insulated along its sides but not at its ends, which initially is at temperature 25. Suddenly both ends are plunged into a bath of temperature 0.

(a) Write the partial differential equation, boundary conditions, and initial condition that govern the temperature of the rod.

(b) Find a formula for the temperature \( u(x,t) \) of the rod at position \( x \) in \([0,1]\) and at time \( t \geq 0 \). You may assume that

\[
25 = \frac{100}{\pi} \left[ \sin(\pi x) + \frac{1}{3} \sin(3\pi x) + \frac{1}{5} \sin(5\pi x) + \cdots \right] \quad \text{for} \quad 0 < x < 1.
\]

(a) \( u_t - k u_{xx} = 0 \) in \( 0 < x < 1, \ 0 < t < \infty \),

\( \text{(b)} \)

\[ u(0,t) = u(1,t) \quad \text{for} \quad t \geq 0, \]

\( \text{(c)} \)

\[ u(x,0) = 25 \quad \text{for} \quad 0 < x < 1. \]

\( \text{Dirichlet B.C.'s.} \)

\( \text{Eigenvalues:} \quad \lambda_n = (n\pi)^2 \quad \text{for} \quad n = 1, 2, 3, \ldots \)

\( \text{Eigenfunctions:} \quad \phi_n(x) = \sin(n\pi x) \)

\[ \text{Therefore} \quad T_n(t) = e^{-k\lambda_n^2 t} \quad \text{solves} \quad \text{\textcircled{1-3-3}.} \]

\[ u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-k \lambda_n^2 t} \quad \text{solves} \quad \text{\textcircled{1-3-3}.} \]

\[ 25 = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \quad \text{for all} \quad 0 < x < 1. \]

By inspection (and comparing with identity \((\dagger)\) above) we see that

\[ b_n = \begin{cases} \frac{100}{\pi (2m+1)} & \text{if} \quad n = 2m+1, \\ 0 & \text{if} \quad n = 2m. \end{cases} \]

Thus

\[ u(x,t) = \frac{100}{\pi} \sum_{m=0}^{\infty} \frac{\sin((2m+1)\pi x) e^{-k(2m+1)\pi^2 t}}{2m+1}. \]
4. (25 pts.) Show that any solution of the damped wave equation

\[ \rho u_{tt} - T u_{xx} + ru_t = 0 \]

(where \( \rho, T, \) and \( r \) are positive constants) has an energy function

\[ E(t) = \frac{1}{2} \int_{-\infty}^{\infty} \left[ \rho u_t^2(x,t) + T u_x^2(x,t) \right] dx \]

that is nonincreasing.

\[ \frac{dE}{dt} = \frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} \left[ \rho u_t^2(x,t) + T u_x^2(x,t) \right] dx \]

\[ = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d}{dt} \left[ \rho u_t^2(x,t) + T u_x^2(x,t) \right] dx \]

\[ = \int_{-\infty}^{\infty} \left[ \rho u_t(x,t)u_{tt}(x,t) + T u_x(x,t)u_{xt}(x,t) \right] dx \]

\[ = \int_{-\infty}^{\infty} \left[ u_t(x,t) \left( T u_x(x,t) - ru_t(x,t) \right) + Tu_x(x,t)u_{xt}(x,t) \right] dx \]

\[ = -r \int_{-\infty}^{\infty} u_t^2(x,t) dx + T \int_{-\infty}^{\infty} \frac{d}{dx} \left( u_t(x,t)u_x(x,t) \right) dx \]

\[ = -r \int_{-\infty}^{\infty} u_t^2(x,t) dx + T \left[ \lim_{x \to -\infty} u_t(x,t)u_x(x,t) - \lim_{x \to \infty} u_t(x,t)u_x(x,t) \right]. \]

Assuming that \( \lim_{|x| \to \infty} u_t(x,t)u_x(x,t) = 0 \) for all real \( t \), we have

\[ \frac{dE}{dt} = -r \int_{-\infty}^{\infty} u_t^2(x,t) dx \leq 0. \]

Thus \( E \) is a nonincreasing function of \( t \).