3. (26 pts.) Consider the 2π-periodic function \( f \) given on one period by 
\[ f(x) = x^2 \text{ if } -\pi \leq x < \pi. \]

(a) Calculate the full Fourier series of \( f \) on \([-\pi, \pi]\).

(b) Write the sum of the first three nonzero terms of the full Fourier series of \( f \) and sketch the graph of this sum on \([-\pi, \pi]\). On the same coordinate axes, sketch the graph of \( f \).

(c) Does the full Fourier series of \( f \) converge to \( f \) in the mean square sense on \([-\pi, \pi]? \) Why?

(d) Does the full Fourier series of \( f \) converge to \( f \) pointwise on \([-\pi, \pi]? \) Why?

(e) Does the full Fourier series of \( f \) converge to \( f \) uniformly on \([-\pi, \pi]? \) Why?

(f) Use the results above to help find the sum 
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}. \]

(g) Use the results above to help find the sum 
\[ \sum_{n=1}^{\infty} \frac{1}{n^4}. \]

(a) Since \( f \) is even, the full Fourier series of \( f \) is a cosine series, i.e. \( b_n = 0 \) for all \( n \geq 1 \).

For \( n \geq 1 \), 
\[ a_n = \frac{\langle f, \cos(nx) \rangle}{\langle \cos(nx), \cos(nx) \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) \, dx = \frac{2}{\pi} \int_{0}^{\pi} x^2 \cos(nx) \, dx = \frac{2}{\pi} \left[ \frac{x^2 \frac{\sin(nx)}{n}}{0} \right]_{0}^{\pi} - \frac{2}{\pi} \int_{0}^{\pi} \frac{2x \sin(nx)}{n} \, dx = \frac{4}{\pi n^2} \int_{0}^{\pi} x \sin(nx) \, dx = \frac{4}{\pi^2 n^2} \int_{0}^{\pi} \frac{x^2 \sin(nx)}{n} \, dx = \frac{4}{\pi^2 n^2} \left[ \frac{x \cos(nx)}{n} \right]_{0}^{\pi} = \frac{4(-1)^n}{n^2}. \]

\[ a_0 = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{1}{\pi} \int_{0}^{\pi} x^2 \, dx = \frac{\pi^2}{3}. \]

\[ f(x) \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n \cos(nx)}{n^2}. \]

(b) \( S(x) = \frac{\pi^2}{3} - 4 \cos(x) + \cos(2x) \)
(c) Since \( \int_{-\pi}^{\pi} f(x) \, dx = \int_{-\pi}^{\pi} (x^5) \, dx = \frac{2\pi^5}{5} < \infty \), the \( L^2 \) convergence theorem (3) implies the full Fourier series of \( f \) converges to \( f \) in the mean square sense on \((-\pi, \pi)\).

(d) Since \( f \) is continuous and \( f \) is piecewise continuous and both are \( 2\pi \)-periodic, Theorem 4 implies the full Fourier series of \( f \) converges pointwise to \( f(x) \) for all \( x \in (-\infty, \infty) \).

(e) Yes, the full Fourier series of \( f \) converges uniformly to \( f \) on \([-\pi, \pi]\), although the uniform convergence theorem (2) doesn’t apply. (The function \( f \) does not satisfy the second periodic boundary condition \( \varphi(-\pi) = \varphi(\pi) \).) To see this,

\[
\max_{-\pi \leq x \leq \pi} \left| f(x) - \frac{\pi^2}{3} - \sum_{n=1}^{N} \frac{4(-1)^n \cos(nx)}{n^2} \right| \leq \max_{-\pi \leq x \leq \pi} \left| \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n \cos(nx)}{n^2} \right|
\]

\[
= \max_{-\pi \leq x \leq \pi} \left| \sum_{n=N+1}^{\infty} \frac{4(-1)^n \cos(nx)}{n^2} \right| \leq \sum_{n=N+1}^{\infty} \frac{4}{n^2} \to 0 \text{ as } N \to \infty \quad \text{(since it is the “tail” of the series \( \sum \frac{1}{n^2} \) which converges by the p-series test with } p=2 \).}

\[
(f) \quad 0 = f(0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n \cos(0)}{n^2}, \quad \text{so} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{\pi^2}{6} = \frac{-\pi^2}{12}
\]

\[
(g) \quad \text{By Parseval’s identity } \sum_{m=0}^{\infty} |A_m|^2 \int_{-\pi}^{\pi} |\widehat{f}(x)|^2 \, dx = \int_{-\pi}^{\pi} |f(x)|^2 \, dx, \quad \text{we have}
\]

\[
\int_{-\pi}^{\pi} \frac{1}{3} \int_{-\pi}^{\pi} \left\| \frac{4(-1)^n \cos(nx)}{n^2} \right\|^2 \, dx = \int_{-\pi}^{\pi} \left| f(x) \right|^2 \, dx
\]

\[
\Rightarrow 2\pi^5 + \sum_{n=1}^{\infty} \frac{16}{n^2} \cdot \pi = \frac{2\pi^5}{9}
\]

\[
\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \left( \frac{2\pi^5}{9} - \frac{2\pi^5}{5} \right) \frac{1}{16\pi} = \frac{8\pi^5}{45} \cdot \frac{1}{16\pi} = \frac{\pi^4}{90}
\]
4. (26 pts.) Let \( u \) be the solution to the problem

\[ \nabla^2 u = 0 \text{ in the disk } \mathcal{D} = \{ (r; \theta) : 0 \leq r < 2, -\pi \leq \theta < \pi \}, \]

\[ u(2; \theta) = 3\sin(2\theta) + 1 \text{ for } -\pi \leq \theta < \pi. \]

(a) Find the maximum value of \( u \) in

\[ \overline{\mathcal{D}} = \{ (r; \theta) : 0 \leq r \leq 2, -\pi \leq \theta < \pi \}. \]

(b) Calculate the value of \( u \) at the origin.

(Hint: These problems can be answered without computing an explicit formula for \( u \) as a function of \( r \) and \( \theta \).)

(a) By the maximum principle, the maximum value of \( u \) occurs on the circumference of the disk: \( \partial \mathcal{D} = \{ (2; \theta) : -\pi \leq \theta < \pi \} \). For all \( \theta \in [-\pi, \pi) \) we have

\[ u(2; \theta) = 3\sin(2\theta) + 1 \leq 3\sin(2(\frac{\pi}{2})) + 1 = 3\sin(\pi) + 1 = 3 \cdot 1 + 1 = 4. \]

(b) By the mean-value theorem, the value of \( u \) at the origin is equal to the average value of \( u \) on the circumference of the disk, \( \partial \mathcal{D} \). Thus

\[ u(0; \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(2; \theta) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} (3\sin(2\theta) + 1) d\theta = \frac{1}{2\pi} \left[ \frac{3\cos(2\theta)}{2} + \theta \right]_{-\pi}^{\pi} = 1. \]
5. (26 pts.) Find the steady-state temperature distribution inside an annular plate with inner radius 1 and outer radius 2 if the outer edge \( r = 2 \) is insulated and on the inner edge \( r = 1 \) the temperature is maintained as \( \theta^2 \) for \( -\pi \leq \theta < \pi \). (Hint: You should find the results of problem 3 useful.)

We need to solve

\[
\begin{align*}
\text{(1)} & \quad \nabla^2 u = 0 \quad \text{in} \quad A = \{(r, \theta) : 1 < r < 2, -\pi \leq \theta < \pi \}, \\
\text{(2)} & \quad u_r(r, \theta) = 0 \quad \text{for} \quad -\pi \leq \theta < \pi, \\
\text{(3)} & \quad u(1, \theta) = \theta^2 \quad \text{for} \quad -\pi \leq \theta < \pi;
\end{align*}
\]

we also have the simplified boundary conditions \( u(r, -\pi) = u(r, \pi) \) and \( u_\theta(r, -\pi) = u_\theta(r, \pi) \) for \( 1 < r < 2 \).

We seek nontrivial solutions to the homogeneous part of the problem \( \text{(1)-(2)-(3)-(5)} \) of the form

\[ u(r, \theta) = R(r) \Theta(\theta). \]

Substituting in \( \text{(1)-(2)-(3)-(5)} \) and simplifying yields

\[ 0 = \nabla^2 u = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = R''(r) \Theta(\theta) + \frac{1}{r} R'(r) \Theta'(\theta) + \frac{1}{r^2} R(r) \Theta''(\theta) \]

\[ \Rightarrow \quad \frac{r^2 R''(r) + r R'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta'(\theta)} = \lambda \]

\[ 0 = u_r(r, \theta) = R'(r) \Theta(\theta) \quad \text{for} \quad -\pi \leq \theta < \pi, \quad 0 = u(1, \theta) - u(\pi, \theta) = R(r) [\Theta'(\pi) - \Theta'(\pi)] \]

\[ 0 = u_\theta(r, \theta) - u_\theta(\pi, \theta) = R(r) [\Theta'(-\pi) - \Theta'(-\pi)] \]

Thus

\[
\begin{bmatrix}
\Theta''(\theta) + \lambda \Theta(\theta) = 0, \\
\Theta'(\pi) = \Theta'(-\pi), \\
\Theta'(\pi) = \Theta'(\pi)
\end{bmatrix}
\]

Eigenvalue Problem

\[ r^2 R''(r) + r R'(r) - \lambda R(r) = 0, \quad R'(2) = 0 \]

The eigenvalues/eigenfunctions are

\[ \lambda_0 = 0, \quad \Theta_0(\theta) = a_0, \]

\[ \lambda_n = n^2, \quad \Theta_n(\theta) = a_n \cos(n \theta) + b_n \sin(n \theta) \quad (n = 1, 2, 3, \ldots) \]

and a solution to the radial problem \( r^2 R''(r) + r R'(r) - n^2 R(r) = 0 \) is of the form \( R_n(r) = r^\alpha \) where \( \alpha \) is a constant. Then \( R_n'(r) = \alpha r^{\alpha-1}, R_n''(r) = \alpha(\alpha-1) r^{\alpha-2} \),

\[ 2 \alpha(r^{\alpha-2} + r \alpha r^{\alpha-1} - n^2 r^\alpha) = 0 \quad \Rightarrow \quad \alpha(\alpha-1) + \alpha - n^2 = 0 \quad \Rightarrow \quad \alpha = \pm n. \]

If \( n \geq 1 \), then \( R_n(r) = c r^n + d r^{-n} \) and \( 0 = R_n(2) \) imply \( \alpha = n \) or \( \alpha = -n \),

\[ 0 = n c 2^n + d 2^{-n} \]

\[ 0 = n c 2^n - d 2^{-n} \]
so \( d = c2^{2n} \); i.e. \( R_n(r) = r^n + c2^n r^{-n} \) (up to a constant factor).

If \( n = 0 \) then the general solution to \( r^2 R''_0(r) + r R'_0(r) = 0 \) is found to follow: 
\( (r R'(r))' = r R''_0(r) + R'_0(r) = 0 \Rightarrow r R'(r) = c \Rightarrow R_0(r) = \int \frac{c}{r} dr = c \ln(r) + d \). Then \( 0 = R'_0(2) = \frac{c}{2} \Rightarrow R_0(r) = 1 \) (up to a constant factor).

A formal solution to (1 - 2 - 3 - 4 - 5) is 
\[ u(r; \theta) = \sum_{n=0}^{\infty} R_n(r) Q_n(\theta) = a_o + \sum_{n=1}^{\infty} (r^n + 2^n r^{-n})(a_n \cos(n\theta) + b_n \sin(n\theta)). \]

We want to choose the arbitrary coefficients so (3) is satisfied:
\[ \theta^2 = u(1; \theta) = a_o + \sum_{n=1}^{\infty} (2n + 1)(a_n \cos(n\theta) + b_n \sin(n\theta)) \quad \text{for } -\pi \leq \theta \leq \pi. \]

By problem #3 (d),
\[ \theta^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n \cos(n\theta)}{n^2} \quad \text{for all } -\pi \leq \theta \leq \pi. \]

Therefore \( a_o = \frac{\pi^2}{3} \), 
\[ \frac{4(-1)^n}{n^2} = (2n + 1)a_n \], and \( b_n = 0 \) for \( n \geq 1 \).

Consequently,
\[ u(r; \theta) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n \cos(n\theta)(r^n + 2^n r^{-n})}{n^2 (1 + 2^n)}. \]
1. (10 pts.) (a) Let \( n \) be a nonnegative integer. Show that the operator \( T \) given by
\[
Tf(r) = \frac{1}{r} \frac{d}{dr} \left( \frac{df}{dr} \right) - \frac{n^2}{r^2} f(r) \quad (0 < r \leq 1)
\]
is hermitian on the vector space
\[
V_B = \{ f \in C^2(0,1) : f(1) = 0, f \text{ and } f' \text{ bounded on } (0,1) \}
\]
equipped with the inner product
\[
<f,g> = \int_0^1 f(r)g(r)rdr.
\]
(b) Are the eigenvalues of \( T \) on \( V_B \) real numbers?
(c) Are the eigenvalues of \( T \) on \( V_B \) positive?
(d) Are the eigenfunctions of \( T \) on \( V_B \), corresponding to distinct eigenvalues, orthogonal on \( (0,1) \) relative to the inner product (*)?

(Please give reasons for your answers to (b)-(d).)

(10 pts.) Use separation of variables to solve the variable density vibrating string problem:
\[
\frac{1}{(1+x)^2} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for } 0 < x < 1, \ 0 < t < \infty,
\]
\[
u(0,t) = 0 = u(1,t) \quad \text{for } 0 \leq t < \infty,
\]
\[
u(x,0) = x(1-x)\sqrt{1+x} \quad \text{and} \quad u_t(x,0) = 0 \quad \text{for } 0 \leq x \leq 1.
\]