Mathematics 325 Exam III Summer 2008 Name: Dr. Grow

1.(40 pts.) 8 (a) Show that the full Fourier series of $f(x) = x^3 - x$ on [-1,1] is $\sum_{n=1}^{\infty} \frac{12(-1)^n \sin(n\pi x)}{(\pi n)^3}$.

5 (b) On the same coordinate axes, sketch the graph of f and the sum of the first three nonzero terms of the full Fourier series of f on [-1,1].

5 (c) Discuss the L^2 – convergence, or lack thereof, for the full Fourier series of f on [-1,1].

5 (d) Discuss the pointwise convergence, or lack thereof, for the full Fourier series of f on [-1,1].

7 (e) Discuss the uniform convergence, or lack thereof, for the full Fourier series of f on [-1,1].

5 (f) Use the results above to help compute the sum of the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}$.

5 (g) Use the results above to help compute the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^6}$.

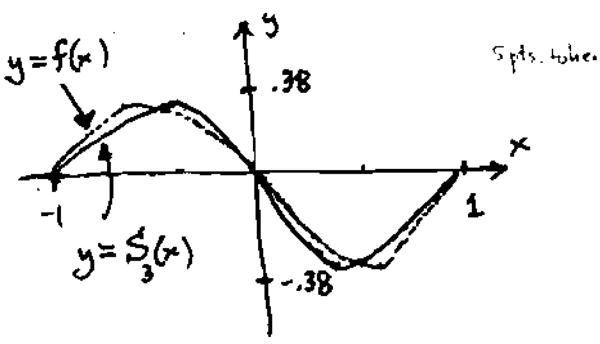
(a) $f(x) \sim a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\pi x) + b_n \sin(n\pi x)\right]$ where $a_0 = \frac{1}{2}\int_{-1}^{1} \overbrace{f(x)}^{\text{odd}}\,dx = 0$, 1 pt. to here

2 pts. to here $a_n = \int_{-1}^{1} \underbrace{\overbrace{f(x)}^{\text{odd}}\overbrace{\cos(n\pi x)}^{\text{even}}}_{\text{odd}}\,dx = 0$, $b_n = \int_{-1}^{1} \underbrace{\overbrace{f(x)}^{\text{odd}}\overbrace{\sin(n\pi x)}^{\text{odd}}}_{\text{even}}\,dx = 2\int_0^1 \underbrace{(x^3-x)}_{v}\underbrace{\sin(n\pi x)\,dx}_{dV} =$ 3 pts. to here.

4 pts. to here $\cancel{2(x^3-x)\left(\frac{-\cos(n\pi x)}{n\pi}\right)\Big|_0^1}^{\,0} + \frac{2}{n\pi}\int_0^1 \underbrace{(3x^2-1)}_{v}\underbrace{\cos(n\pi x)\,dx}_{dV} = \frac{2}{n\pi}(3x^2-1)\frac{\sin(n\pi x)}{n\pi}\Big|_0^1 - \frac{12}{(n\pi)^2}\int_0^1 \overbrace{x}^{v}\,\overbrace{\sin(n\pi x)\,dx}^{dV}$ 5 pts. to here.

6 pts. to here. $= \frac{-12}{(n\pi)^2}\, x\left(-\frac{\cos(n\pi x)}{n\pi}\right)\Big|_0^1 + \frac{12}{(n\pi)^3}\cancel{\int_0^1 \cos(n\pi x)\,dx}^{\,0} = \frac{12\cos(n\pi)}{(n\pi)^3} = \frac{12(-1)^n}{(n\pi)^3}$. 7 pts. to here

8 pts. to here Thus $\boxed{f(x) \sim \sum_{n=1}^{\infty} \frac{12(-1)^n \sin(n\pi x)}{(n\pi)^3}.}$



5 pts. to here

(b) $S_3(x) = \sum_{n=1}^{3} \frac{12(-1)^n \sin(n\pi x)}{(n\pi)^3}$

$\Rightarrow \boxed{S_3(x) = \frac{12}{\pi^3}\left(-\sin(\pi x) + \frac{1}{8}\sin(2\pi x) - \frac{1}{27}\sin(3\pi x)\right)}$

3 pts. to here.

(c) The full Fourier series of f is the Fourier series of f with respect to the orthogonal set of functions $\Phi = \{1, \cos(\pi x), \sin(\pi x), \cos(2\pi x), \sin(2\pi x), \ldots\}$ on the interval $(-1,1)$. These functions are the complete set of eigenfunctions for the problem

$X''(x) + \lambda X(x) = 0, \quad X(1) = X(-1), \quad X'(1) = X'(-1),$

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Note that $f(x) = x^3 - x$ satisfies the (periodic) boundary conditions:

3 pts. to here. $f(1) = 0 = f(-1)$ and $f'(1) = 2 = f'(-1)$. Furthermore $f(x) = x^3 - x$, $f'(x) = 3x^2 - 1$,

5 pts. to here. and $f''(x) = 6x$ are continuous on the closed interval $-1 \le x \le 1$. Consequently, Theorem 2 guarantees that the full Fourier series of f converges uniformly to f

7 pts. to here

on $[-1,1]$.

(c)-(d). Because uniform convergence to f on $[-1,1]$ implies both L^2-convergence

10 pts.

and pointwise convergence to f on $[-1,1]$, it follows that the full Fourier series of f converges to f, both in the mean-square sense and the pointwise sense, on $[-1,1]$.

n	$\sin(n\pi/2)$
1	1
2	0
3	-1
4	0
⋮	⋮
$2k$	0
$2k+1$	$(-1)^k$

(f) $$\left(\tfrac{1}{2}\right)^3 - \tfrac{1}{2} = f\left(\tfrac{1}{2}\right) \overset{\text{by part (d)}}{=} \sum_{n=1}^{\infty} \frac{12(-1)^n \sin(n\pi/2)}{(n\pi)^3}$$ 3 pts. to here.

$$\Rightarrow -\frac{3}{8} = \frac{12}{\pi^3} \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}(-1)^k}{(2k+1)^3}$$

$$\Rightarrow \left(-\frac{\pi^3}{\cancel{12}_{4}}\right)\left(-\frac{\cancel{3}}{8}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}$$

5 pts. to here.

$$\boxed{\frac{\pi^3}{32} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}}$$

(g) By Parseval's identity, $\displaystyle\sum_{n=1}^{\infty} |c_n|^2 \int_a^b |X_n(x)|^2\,dx = \int_a^b |f(x)|^2\,dx$ 2 pts. to here

when $\{X_1, X_2, X_3, \dots\}$ is a complete set of orthogonal functions on $[a,b]$. Applying this to our case, $\displaystyle\sum_{n=1}^{\infty} |b_n|^2 \int_{-1}^{1} \sin^2(n\pi x)\,dx = \int_{-1}^{1} |x^3 - x|^2\,dx$ (since $a_0 = a_1 = \dots = 0$). Thus $\displaystyle\sum_{n=1}^{\infty} \left|\frac{12(-1)^n}{(n\pi)^3}\right|^2 \cdot 1 = 2\int_0^1 (x - x^3)^2\,dx = \frac{16}{105}$,

so $$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{16}{105}\cdot\frac{\pi^6}{[illegible]} = \frac{\pi^6}{[illegible]}}$$ 5 pts. to here

Please recall that the Laplacian of u in polar coordinates is $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$ while in spherical coordinates it is $\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin(\phi)}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial u}{\partial \phi}\right) + \frac{1}{r^2\sin^2(\phi)}\frac{\partial^2 u}{\partial \theta^2}$.

2.(30 pts.) Let $0 < a < b < \infty$. Solve $\nabla^2 u = 1$ in $\mathcal{R}: a^2 < x^2 + y^2 + z^2 < b^2$ subject to $u = 0$ on $\partial\mathcal{R}$.

3 pts. to here. Assume $u = u(r)$, independent of the spherical angles φ and θ. (This is reasonable to assume because the PDE, region, and boundary conditions are invariant with respect to rotations.) Then $\frac{\partial u}{\partial \varphi} = \frac{\partial^2 u}{\partial \theta^2} = 0$ so $\nabla^2 u = 1$ becomes $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = 1$ 9 p to he

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = r^2 \Rightarrow r^2\frac{\partial u}{\partial r} = \frac{r^3}{3} + c_1 \Rightarrow \frac{\partial u}{\partial r} = \frac{r}{3} + \frac{c_1}{r^2} \Rightarrow$$

15 pts. to here

21 pts. to here. $u = \frac{r^2}{6} - \frac{c_1}{r} + c_2$. Applying the boundary conditions $u\big|_{r=a} = 0 = u\big|_{r=b}$ we have

25 pts. to here.

$$\left.\begin{aligned} 0 &= \frac{a^2}{6} - \frac{c_1}{a} + c_2 \\ 0 &= \frac{b^2}{6} - \frac{c_1}{b} + c_2 \end{aligned}\right\} \text{ subtract : } 0 = \frac{b^2 - a^2}{6} - c_1\left(\frac{1}{b} - \frac{1}{a}\right) \Rightarrow -c_1\left(\frac{b-a}{ab}\right) = \frac{b^2 - a^2}{6}$$

$$\therefore c_1 = \frac{-(b+a)ab}{6}$$

27 pts. to here.

$$\therefore c_2 = \frac{c_1}{a} - \frac{a^2}{6} = \frac{-(b+a)b}{6} - \frac{a^2}{6}.$$

29 pts. to here

Thus $\boxed{u(r,\theta,\varphi) = \frac{r^2}{6} + \frac{(b+a)ab}{6r} - \frac{(b+a)b}{6} - \frac{a^2}{6}}$. 30 pts to here

This simplifies to $\boxed{u(r,\theta,\varphi) = \frac{(r-a)(r-b)(r+a+b)}{6r}}$.

Note: The solution above is unique. This is a direct consequence of the maximum/minimum principle for harmonic functions.

3.(30 pts.) Solve $\nabla^2 u = 0$ in the cube $0 < x < 1$, $0 < y < 1$, $0 < z < 1$ given that u satisfies the inhomogeneous **Dirichlet** condition $u(x,0,z) = 4\sin^2(\pi x)\sin^2(\pi z)$ if $0 \le x \le 1$, $0 \le z \le 1$, and u satisfies homogeneous **Neumann** boundary conditions on the other five faces. Hint: You may find the identity $2\sin^2(\theta) = 1 - \cos(2\theta)$ useful.

2 pts. to here. $u(x,y,z) = X(x)Y(y)Z(z)$ (nontrivial solution to the homogeneous portion of this problem) leads to $X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z) = 0$ (when substituted in the PDE)

$$\Rightarrow \quad -\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = \lambda \quad \text{and} \quad -\frac{Z''(z)}{Z(z)} = \frac{Y''(y)}{Y(y)} - \lambda = \mu. \text{ Thus}$$

$$(*) \begin{cases} X''(x) + \lambda X(x) = 0, \quad X'(0) = 0 = X'(1). \\ Z''(z) + \mu Z(z) = 0, \quad Z'(0) = 0 = Z'(1). \\ Y''(y) - (\lambda+\mu)Y(y) = 0, \quad Y'(1) = 0. \end{cases}$$

5 pts. to here

8 pts to here

10 pts to here

[The Neumann boundary conditions $u_x(0,y,z) = u_x(1,y,z) = 0$ for all $0 \le y \le 1, 0 \le z \le 1$, $u_z(x,y,0) = u_z(x,y,1) = 0$ for all $0 \le x \le 1, 0 \le y \le 1$, and $u_y(x,1,z) = 0$ for all $0 \le x \le 1, 0 \le z$ yield the boundary conditions $X'(0) = 0 = X'(1)$, $Z'(0) = Z'(1) = 0$, and $Y'(1) = 0$ above.]

The eigenvalue problems in the first two lines of (*) have solutions:

$$\lambda_\ell = (\ell\pi)^2, \quad X_\ell(x) = \cos(\ell\pi x) \quad (\ell = 0,1,2,\dots), \quad \text{14 pts. to here.}$$

$$\mu_m = (m\pi)^2, \quad Z_m(z) = \cos(m\pi z) \quad (m = 0,1,2,\dots). \quad \text{18 pts. to here}$$

Substituting $\lambda + \mu = \lambda_\ell + \mu_m = \pi^2(\ell^2 + m^2)$ in the third line of (*) and solving yields $Y_{\ell,m}(y) = \cosh\left(\pi(y-1)\sqrt{\ell^2+m^2}\right)$ (up to a constant factor).

20 pts to here

Thus, the superposition principle implies that

$$u(x,y,z) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} A_{\ell,m}\cos(\ell\pi x)\cos(m\pi z)\cosh\left(\pi(y-1)\sqrt{\ell^2+m^2}\right)$$

22 p to he

is a formal solution to the homogeneous portion of this problem for any

(OVER)

choice of constants $A_{\ell,m}$ $(\ell, m = 0, 1, 2, \dots)$. Therefore, to satisfy the inhomogeneous Dirichlet boundary condition, we require that

$$\sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} A_{\ell,m} \cosh\left(-\pi\sqrt{\ell^2+m^2}\right)\cos(\ell\pi x)\cos(m\pi z) = u(x,0,z) = 4\sin^2(\pi x)\sin^2(\pi z)$$

$$= [1-\cos(2\pi x)][1-\cos(2\pi z)]$$

$$= 1 - \cos(2\pi x) - \cos(2\pi z) + \cos(2\pi x)\cos(2\pi z)$$

for all $0 \le x \le 1$ and $0 \le z \le 1$. Consequently, equating like coefficients produces

$$\left.\begin{aligned} A_{0,0} &= 1 \\ A_{2,0}\cosh(-2\pi) &= -1 \\ A_{0,2}\cosh(-2\pi) &= -1 \\ A_{2,2}\cosh\left(-2\pi\sqrt{2}\right) &= 1 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} A_{0,0} &= 1 \\ A_{2,0} &= \frac{-1}{\cosh(2\pi)} = A_{0,2} \\ A_{2,2} &= \frac{1}{\cosh(2\pi\sqrt{2})} \end{aligned}$$

6 pts. to here.

and all other $A_{\ell,m} = 0$. That is,

$$\boxed{u(x,y,z) = 1 - \frac{\cos(2\pi x)\cosh(2\pi(y-1))}{\cosh(2\pi)} - \frac{\cos(2\pi z)\cosh(2\pi(y-1))}{\cosh(2\pi)} + \frac{\cos(2\pi x)\cos(2\pi z)\cosh(2\pi(y-1)\sqrt{2})}{\cosh(2\pi\sqrt{2})}}$$

3 pts to here.

solves the problem.

Bonus (30 pts.). Solve $u_t - u_{xx} \overset{(1)}{=} 0$ in $-1 < x < 1$, $0 < t < \infty$, subject to the boundary conditions $u(1,t) \overset{(2)}{=} u(-1,t)$ and $u_x(1,t) \overset{(3)}{=} u_x(-1,t)$ if $t \geq 0$, and the initial condition $u(x,0) \overset{(4)}{=} x^3 - x$ if $-1 \leq x \leq 1$. Furthermore, show that this solution is unique. Hint: You may find the results of problem 1 useful.

2 pts. to here. We seek nontrivial solutions of (1)-(2)-(3) of the form $u(x,t) = X(x)T(t)$. Substituting into (1) yields $T'(t)X(x) - T(t)X''(x) = 0 \Rightarrow \dfrac{T'(t)}{T(t)} = \dfrac{X''(x)}{X(x)} = -\lambda$. Substituting in (2)-(3) yields $X(1)T(t) = X(-1)T(t)$ and $X'(1)T(t) = X'(-1)T(t)$ for all $t \geq 0$. Using the fact that $u(x,t) = X(x)T(t)$ is not identically zero yields

6 pts. to here.

$$(*) \quad \begin{cases} X''(x) + \lambda X(x) = 0, \quad X(1) = X(-1), \quad X'(1) = X'(-1), \\ T'(t) + \lambda T(t) = 0. \end{cases}$$

7 pts. to here.

pts. to here. The eigenvalue problem in the first line of $(*)$ has solutions $\lambda = \lambda_n = (n\pi)^2$ and

11 pts. to here. $X_n(x) = a_n \cos(n\pi x) + b_n \sin(n\pi x)$ $\quad$ ($n = 0, 1, 2, 3, \ldots$ and a_n, b_n arbitrary constants).

13 pts. to here. Setting $\lambda = \lambda_n = (n\pi)^2$ in the second line of $(*)$ yields $T_n(t) = e^{-(n\pi)^2 t}$ (up to a

15 pts. to here. constant factor). Thus $u(x,t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\pi x) + b_n \sin(n\pi x)\right] e^{-(n\pi)^2 t}$

is a formal solution to (1)-(2)-(3) for any choice of constants $a_0, a_1, b_1, a_2, b_2, \ldots$. We want to choose these constants so the initial condition (4) is met. Using the full Fourier series for $f(x) = x^3 - x$ in $[-1,1]$ in problem 1 yields

17 pts. to here.

$$a_0 + \sum_{n=0}^{\infty} \left[a_n \cos(n\pi x) + b_n \sin(n\pi x)\right] = u(x,0) = x^3 - x = \sum_{n=1}^{\infty} \frac{12(-1)^n \sin(n\pi x)}{(\pi n)^3}$$

19 pts. to here. for all $-1 \leq x \leq 1$. Therefore choose $a_0 = a_1 = a_2 = \ldots = 0$ and $b_n = \dfrac{12(-1)^n}{(\pi n)^3}$ for $n = 1, 2, 3, \ldots$

That is,

21 pts. to here.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{12(-1)^n \sin(n\pi x) e^{-(n\pi)^2 t}}{(\pi n)^3}}.$$

To show that this solution is unique, suppose that $v = v(x,t)$ is any other

(OVER)

22 pts. to here. solution of ①-②-③-④ and consider $w(x,t) = u(x,t) - v(x,t)$. Then w satisfies

$$(1') \qquad w_t - w_{xx} = 0 \quad \text{if} \quad -1 < x < 1,\ 0 < t < \infty,$$

24 pts. to here.

$$(2')\text{-}(3') \qquad w(1,t) = w(-1,t) \quad \text{and} \quad w_x(1,t) = w_x(-1,t) \quad \text{if} \quad t \geq 0,$$

$$(4') \qquad w(x,0) = 0 \quad \text{if} \quad -1 \leq x \leq 1 .$$

26 pts. to here. Consider the energy $E(t) = \int_{-1}^{1} w^2(x,t)\,dx$ of this solution at $t \geq 0$.

Then
$$\frac{dE}{dt} = \int_{-1}^{1} \frac{\partial}{\partial t}\left[w^2(x,t)\right] dx = \int_{-1}^{1} 2w(x,t)\,w_t(x,t)\,dx = \int_{-1}^{1} \underbrace{2w(x,t)}_{U}\,\underbrace{w_{xx}(x,t)\,dx}_{dV}$$

$$= 2w(x,t)w_x(x,t)\Big|_{x=-1}^{1} - \int_{-1}^{1} 2\,w_x^2(x,t)\,dx$$

$$= \underbrace{2\,w(1,t)w_x(1,t) - 2w(-1,t)w_x(-1,t)}_{0 \text{ by } (2')\text{-}(3')} - 2\int_{-1}^{1} w_x^2(x,t)\,dx$$

28 pts. to here.
$$= -2\int_{-1}^{1} w_x^2(x,t)\,dx \quad \leq 0 \quad . \quad (\text{I.e. } E = E(t) \text{ is a decreasing function.})$$

Therefore $0 \leq E(t) \leq E(0) = \int_{-1}^{1} w^2(x,0)\,dx \overset{\text{by } (4')}{=} \int_{-1}^{1} 0\,dx = 0$ for $t \geq 0$. By the vanishing theorem, the nonnegative continuous integrand of $E(t)$ must be identically zero for all $-1 \leq x \leq 1$ and each $t \geq 0$. That is,

30 pts. to here.
$$u(x,t) - v(x,t) = w(x,t) = 0 \qquad \text{for all } -1 \leq x \leq 1,\ 0 \leq t < \infty,$$

and hence the solution to ①-②-③-④ is unique.

Convergence Theorems

$$X'' + \lambda X = 0 \text{ in } (a, b) \text{ with any } \textit{symmetric} \text{ BC.} \qquad (1)$$

Now let $f(x)$ be any function defined on $a \le x \le b$. Consider the Fourier series for the problem (1) with any given boundary conditions that are symmetric. We now state a convergence theorem for each of the three modes of convergence. They are partly proved in the next section.

Theorem 2. Uniform Convergence The Fourier series $\Sigma A_n X_n(x)$ converges to $f(x)$ uniformly on $[a, b]$ provided that

(i) $f(x)$, $f'(x)$, and $f''(x)$ exist and are continuous for $a \le x \le b$ and
(ii) $f(x)$ satisfies the given boundary conditions.

Theorem 3. L^2 Convergence The Fourier series converges to $f(x)$ in the mean-square sense in (a, b) provided only that $f(x)$ is any function for which

$$\int_a^b |f(x)|^2\, dx \text{ is finite.} \qquad (8)$$

Theorem 4. Pointwise Convergence of Classical Fourier Series

(i) The classical Fourier series (full or sine or cosine) converges to $f(x)$ pointwise on (a, b), provided that $f(x)$ is a continuous function on $a \le x \le b$ and $f'(x)$ is piecewise continuous on $a \le x \le b$.
(ii) More generally, if $f(x)$ itself is only piecewise continuous on $a \le x \le b$ and $f'(x)$ is also piecewise continuous on $a \le x \le b$, then the classical Fourier series converges at every point x $(-\infty < x < \infty)$. The sum is

$$\sum_n A_n X_n(x) = \tfrac{1}{2}[f(x+) + f(x-)] \qquad \text{for all } a < x < b. \qquad (9)$$

The sum is $\frac{1}{2}[f_{\text{ext}}(x+) + f_{\text{ext}}(x-)]$ for all $-\infty < x < \infty$, where $f_{\text{ext}}(x)$ is the extended function (periodic, odd periodic, or even periodic).

Theorem 4∞. If $f(x)$ is a function of period $2l$ on the line for which $f(x)$ and $f'(x)$ are piecewise continuous, then the classical full Fourier series converges to $\frac{1}{2}[f(x+) + f(x-)]$ for $-\infty < x < \infty$.