Mathematics 325 — Exam I, Fall 2005 — Name: Dr. Grow (5 pts.)

1.(15 pts.) For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous. Provide reasons for your answers.

(a) $\overbrace{u_t - u_{xx} + xu}^{\mathcal{L}(u)} + 1 = 0$ (b) $\overbrace{u_{tt} - u_{xxt} + uu_x}^{\mathcal{L}(u)} = 0$ (c) $\overbrace{u_x + e^y u_y}^{\mathcal{L}(u)} = 0$

5 (a) Second order (1), linear (3), and inhomogeneous (1) $\longrightarrow \mathcal{L}(u) = -1$.

$$\mathcal{L}(u+v) = (u+v)_t - (u+v)_{xx} + x(u+v) = u_t - u_{xx} + xu + v_t - v_{xx} + xv = \mathcal{L}(u) + \mathcal{L}(v)$$

$$\mathcal{L}(ku) = (ku)_t - (ku)_{xx} + x(ku) = k(u_t - u_{xx} + xu) = k\mathcal{L}(u)$$

5 (b) Third order (1), nonlinear (4)

$$\mathcal{L}(ku) = (ku)_{tt} - (ku)_{xxt} + (ku)(ku)_x = k(u_{tt} - u_{xxt} + kuu_x) \neq k\mathcal{L}(u)$$

5 (c) First order (1), linear (3), and homogeneous (1) $\longrightarrow \mathcal{L}(u) = 0$.

$$\mathcal{L}(u+v) = (u+v)_x + e^y(u+v)_y = u_x + e^y u_y + v_x + e^y v_y = \mathcal{L}(u) + \mathcal{L}(v)$$

$$\mathcal{L}(ku) = (ku)_x + e^y(ku)_y = k(u_x + e^y u_y) = k\mathcal{L}(u)$$

2.(20 pts.) Solve the partial differential equation $2yu_x + (4x + xy^2)u_y = 0$ subject to the auxiliary condition $u(0, y) = y^4 - 8y^2$ for $-\infty < y < \infty$.

The pde can be written $\langle 2y, 4x+xy^2 \rangle \cdot \nabla u = 0$. Therefore any solution $u = u(x,y)$ is constant along curves whose tangent lines are parallel to $\langle 2y, 4x+xy^2 \rangle$ at a general

5 pts. point (x,y). I.e. along any curve satisfying $\frac{dy}{dx} = \frac{4x+xy^2}{2y} \Rightarrow \frac{dy}{dx} = \frac{x(4+y^2)}{2y}$

$$\Rightarrow \frac{2y\,dy}{4+y^2} = x\,dx \Rightarrow \ln(4+y^2) = \int \frac{2y\,dy}{4+y^2} = \int x\,dx = \frac{x^2}{2} + c$$

10 pts. $\Rightarrow 4+y^2 = Ae^{x^2/2} \Rightarrow (4+y^2)e^{-x^2/2} = A$. Therefore, along any such curve,

$$u(x,y) = u\left(x, \pm\sqrt{Ae^{x^2/2} - 4}\right) = u\left(0, \pm\sqrt{A-4}\right) = f(A).$$

Therefore, the general solution is

15 pts.

$$u(x,y) = f\left((4+y^2)e^{-x^2/2}\right)$$

where f is any C^1-function of a single real variable.

$$y^4 - 8y^2 = u(0,y) = f(4+y^2) \Rightarrow f(z) = (z-4)^2 - 8(z-4) = z^2 - 8z + 16 - 8z + 32$$

18 pts. (Let $4+y^2 = z$. Then $y^2 = z-4$.) $= z^2 - 16z + 48$

$$\therefore\ u(x,y) = f\left((4+y^2)e^{-x^2/2}\right) = \left[(4+y^2)e^{-x^2/2}\right]^2 - 16(4+y^2)e^{-x^2/2} + 48$$

20 pts.

$$\boxed{u(x,y) = (4+y^2)^2 e^{-x^2} - 16(4+y^2)e^{-x^2/2} + 48}$$

3.(20 pts.) A homogeneous body occupying the solid region $D=\{(x,y,z)\in\mathbf{R}^3 \mid x^2+y^2+z^2\le 4\}$ is completely insulated. Its initial temperature at position (x,y,z) in D is $9(x^2+y^2+z^2)^{3/2}$.

(a) Write the partial differential equation and initial/boundary conditions that model the temperature $u(x,y,z,t)$ of the body at position (x,y,z) and time t. (No derivation is required; merely state the appropriate equations.)

(b) Find the steady-state temperature that the body reaches after a long time. (Hint: No heat is gained or lost by the body.)

9 (a)

$$\begin{cases} u_t - k(u_{xx}+u_{yy}+u_{zz}) = 0 & \text{in } D^\circ : x^2+y^2+z^2<4. \quad (3) \\ u(x,y,z,0) = 9(x^2+y^2+z^2)^{3/2} & \text{in } D : x^2+y^2+z^2 \le 4. \quad (3) \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial D : x^2+y^2+z^2 = 4. \quad (3) \end{cases}$$

11 (b) Since the heat energy of the body is constant, for all $t>0$,

(2) $$\iiint_D c\rho u(x,y,z,t)\,dV = E(t) = E(0) = \iiint_D c\rho u(x,y,z,0)\,dV = \iiint_D c\rho 9(x^2+y^2+z^2)^{3/2}d$$

(2) As $t\to\infty$, $u(x,y,z,t) \longrightarrow \underset{\text{(steady-state temperature)}}{U(x,y,z)} = \text{constant} = U_0$. Therefore

(2) $$E(t) = \iiint_D c\rho u(x,y,z,t)\,dV \longrightarrow \iiint_D c\rho U_0\,dV = c\rho U_0 \,\mathrm{vol}(D), \text{ so}$$

$$c\rho U_0\,\mathrm{vol}(D) = \iiint_D c\rho 9(x^2+y^2+z^2)^{3/2}dV, \text{ and thus } U_0 = \frac{\iiint_D c\rho 9(x^2+y^2+z^2)^{3/2}dV}{c\rho\,\mathrm{vol}(D)} \quad (2)$$

(1) $$= \frac{\int_0^{2\pi}\int_0^{\pi}\int_0^{2} 9r^3\cdot r^2\sin\varphi\,dr\,d\varphi\,d\theta}{\frac{4}{3}\pi(2)^3} = \frac{\left(\frac{\cancel{9}^{3} r^6}{\cancel{6}_{2}}\Big|_0^2\right)\left(-\cos(\varphi)\Big|_0^{\pi}\right)\left(\theta\Big|_0^{2\pi}\right)}{\frac{32\pi}{3}}$$

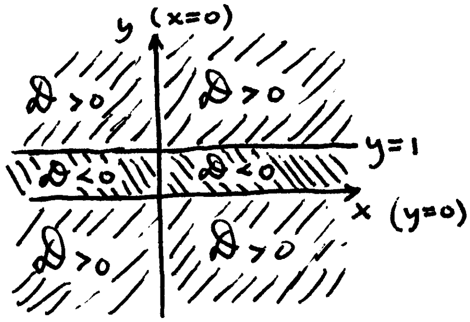
$$= \left(\frac{3}{\cancel{32\pi}}\right)\left(3\cdot\cancel{32}\right)\left(2\right)\left(2\cancel{\pi}\right) = \boxed{36} \quad (2)$$

4.(20 pts.) (a) Find the regions in the xy – plane where the equation $yu_{yy} + 2xyu_{xy} + x^2 u_{xx} = 0$ is elliptic, hyperbolic, or parabolic. Sketch them.

(b) Find the general solution of $u_{xx} - u_{xy} + 3u_{yy} - 3u_{yx} = \sin(x+y)$ in the xy – plane.

10 (a) $\mathcal{D} = B^2 - 4AC = (2xy)^2 - 4(x^2)(y) = 4x^2y^2 - 4x^2y = 4x^2y(y-1)$. Therefore $\mathcal{D} = 0$ (2)

if and only if $x=0$ or $y=0$ or $y=1$.



(3) The pde is elliptic if $\mathcal{D} < 0$. I.e. $\boxed{\text{if } 0 < y < 1 \text{ and } x \neq 0, \text{ the pde is elliptic}}$.

(3) The pde is hyperbolic if $\mathcal{D} > 0$. I.e. $\boxed{\text{if } y > 1 \text{ and } x \neq 0, \text{ or if } y < 0 \text{ and } x \neq 0, \text{ the pde is hyperbolic}}$.

(2) The pde is parabolic if $\mathcal{D} = 0$. I.e. $\boxed{\text{if } x=0 \text{ or } y=0 \text{ or } y=1, \text{ the pde is parabolic}}$.

10 (b) The pde is equivalent to $u_{xx} - 4u_{xy} + 3u_{yy} = \sin(x+y)$ or

(2) $\left(\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u = \sin(x+y)$. Let $\begin{cases} \xi = 3x+y \\ \eta = x+y. \end{cases}$ (2)

Then $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x} = 3\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta} \Rightarrow \frac{\partial}{\partial x} = 3\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$.

Similarly $\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$. Thus $\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y} = \cancel{3\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta} - 3\left(\cancel{\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta}\right) = -2\frac{\partial}{\partial \eta}$

and $\frac{\partial}{\partial x} - \frac{\partial}{\partial y} = 3\frac{\partial}{\partial \xi} + \cancel{\frac{\partial}{\partial \eta}} - \left(\frac{\partial}{\partial \xi} + \cancel{\frac{\partial}{\partial \eta}}\right) = 2\frac{\partial}{\partial \xi}$. Therefore the pde is

(2)

$\left(-2\frac{\partial}{\partial \eta}\right)\left(2\frac{\partial}{\partial \xi}\right)u = \sin(\eta) \Rightarrow \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) = -\frac{1}{4}\sin(\eta) \Rightarrow \frac{\partial u}{\partial \xi} = \int -\frac{1}{4}\sin(\eta)\,d\eta$

$= \frac{1}{4}\cos(\eta) + c(\xi) \Rightarrow u = \int\left[\frac{1}{4}\cos(\eta) + c(\xi)\right]d\xi = \frac{\xi}{4}\cos(\eta) + f(\xi) + g(\eta)$. (2)

The general solution of the pde is $\boxed{u(x,y) = \frac{3x+y}{4}\cos(x+y) + f(3x+y) + g(x+y)}$. (2)

5.(20 pts.) (a) Derive the general solution of the partial differential equation

(*) $$u_{tt} - c^2 u_{xx} = 0$$

in the xt-plane. (Here c is a positive constant.)

(b) Use the answer in part (a) to help derive a formula for the solution to (*) which satisfies the initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$ for $-\infty < x < \infty$. (Here ϕ and ψ are prescribed "sufficiently smooth" functions of a single real variable.)

10 (a) $\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0$. (2) Let $\begin{cases} \xi = ct + x \\ \eta = -ct + x \end{cases}$ (2). Then

$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta} \Rightarrow \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$. Similarly $\frac{\partial}{\partial t} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}$

$\therefore \frac{\partial}{\partial t} - c\frac{\partial}{\partial x} = \cancel{c\frac{\partial}{\partial \xi}} - c\frac{\partial}{\partial \eta} - c\left(\cancel{\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta}\right) = -2c\frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial t} + c\frac{\partial}{\partial x} =$

$c\frac{\partial}{\partial \xi} - \cancel{c\frac{\partial}{\partial \eta}} + c\left(\frac{\partial}{\partial \xi} + \cancel{\frac{\partial}{\partial \eta}}\right) = 2c\frac{\partial}{\partial \xi}$. The pde (*) is equivalent to $\left(-2c\frac{\partial}{\partial \eta}\right)\left(2c\frac{\partial}{\partial \xi}\right)u$

$= 0 \Rightarrow \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) = 0$ (2) $\Rightarrow \frac{\partial u}{\partial \xi} = c(\xi) \Rightarrow u = f(\xi) + g(\eta)$. (2)

$\therefore \boxed{u(x,t) = f(x+ct) + g(x-ct)}$ is the general solution of the pde (*). (2)

10 (b) $f(x) + g(x) = u(x,0) = \varphi(x) \Rightarrow f'(x) + g'(x) = \varphi'(x)$ ① (2)

$u_t(x,t) = cf'(x+ct) - cg'(x-ct) \Rightarrow cf'(x) - cg'(x) = u_t(x,0) = \psi(x)$ ②

$c① + ②$: $2cf'(x) = c\varphi'(x) + \psi(x) \Rightarrow f(x) = \frac{1}{2}\varphi(x) + \frac{1}{2c}\int_0^x \psi(\xi)d\xi + c_1$ ③ (2)

$c① - ②$: $2cg'(x) = c\varphi'(x) - \psi(x) \Rightarrow g(x) = \frac{1}{2}\varphi(x) - \frac{1}{2c}\int_0^x \psi(\xi)d\xi + c_2$ ④ (2

$f(x) + g(x) = \varphi(x) \overset{③\&④}{\Rightarrow} \varphi(x) + c_1 + c_2 = \varphi(x) \Rightarrow c_1 + c_2 = 0$. (1)

$\therefore u(x,t) = f(x+ct) + g(x-ct) = \frac{1}{2}\varphi(x+ct) + \frac{1}{2c}\int_0^{x+ct}\psi(\xi)d\xi + c_1 + \frac{1}{2}\varphi(x-ct) - \frac{1}{2c}\int_0^{x-ct}\psi(\xi)d\xi + c_2$ (

$\Rightarrow u(x,t) = \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\left[\int_0^{x+ct}\psi(\xi)d\xi - \int_0^{x-ct}\psi(\xi)d\xi\right]$

$\Rightarrow \boxed{u(x,t) = \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\xi)d\xi}$ (1)

Math 325

Exam I

Fall 2005

Mean: 62.7

Standard Deviation: 18.2

$n = 21$

Distribution of Scores:

87 – 100	A	2
73 – 86	B	4
60 – 72	B (undergraduate) C (graduate)	6
50 – 59	C	5
0 – 49	D (undergraduate) F (graduate)	4