Mathematics 325 Exam I Summer 2004 Name: Dr. Grow

1.(25 pts.) Consider a function of the form

$u_x = Bt + 2Ax + D$

$$u(x,t) = Ax^2 + Bxt + Ct^2 + Dx + Et + F$$

where A, B, C, D, E, and F are constants.

(a) What is the most general form of u if it solves $u_t - u_{xx} = 0$ in the xt-plane? (Note: The correct answer will involve three arbitrary constants.)

(b) Find the solution of $u_t - u_{xx} = 0$ that satisfies $u(x,0) = 3x^2$ for all $-\infty < x < \infty$.

(a) $0 = u_t - u_{xx} = \overset{4}{(Bx + 2Ct + E)} - \overset{4}{(2A)} = Bx + 2Ct + (E - 2A)$ 3

Therefore $B = 2C = E - 2A = 0$, i.e. $B = 0$, $C = 0$, and $E = 2A$. Thus

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16 pts. $\boxed{u(x,t) = Ax^2 + Dx + 2At + F}$. 1

(b) $3x^2 = u(x,0) \overset{4}{=} Ax^2 + Dx + F$ so $A = 3$, $D = F = 0$. 3

9 pts. Consequently $\boxed{u(x,t) = 3x^2 + 6t}$. 2

2.(25 pts.) Consider the partial differential equation

(*) $$(y+1)u_x + 2xyu_y = 0.$$

(a) Find the characteristic curves of (*).

(b) Write the general solution of (*).

(c) Find the solution of (*) that satisfies the condition $u(0,y) = ye^y$ for $y > 0$.

(a) $\langle y+1, 2xy\rangle \cdot \nabla u = 0$. Therefore u is constant along curves whose tangent line is parallel to $\langle y+1, 2xy\rangle$ at a general point (x,y) on the curve. I.e.

5 3

$$\frac{dy}{dx} = \frac{2xy}{y+1} \Rightarrow \left(\frac{y+1}{y}\right)dy = 2x\,dx \quad \text{(Variables Separable)}$$

$$\Rightarrow \int \left(1+\frac{1}{y}\right)dy = \int 2x\,dx$$

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$$\Rightarrow y + \ln(y) = x^2 + c$$

$$\boxed{y + \ln(y) - x^2 = c}$$ characteristic curves of (*).

12 pts.

(b) Along a characteristic curve $u(x,y) = u(\sqrt{y+\ln(y)-c}, y) \overset{y=1}{=} u(\sqrt{1-c}, 1)$
$= f(c)$.

$\therefore$ $\boxed{u(x,y) = f(y+\ln(y)-x^2)}$ is the general solution of (*).
(f is a C^1-function of a single real variab

8 pts.

(c) $ye^y = u(0,y) = f(y+\ln(y))$ for all $y > 0$. But

$ye^y = e^{\ln(y)} \cdot e^y = e^{y+\ln(y)}$ so $e^{y+\ln(y)} = f(y+\ln(y))$,

3

and hence $f(z) = e^z$ for all real z.

$\therefore$ $\boxed{u(x,y) = e^{y+\ln(y)-x^2}}$ solves the I.V.P.

5 pts.

(This can also be written as $u(x,y) = ye^{y-x^2}$.)

$$\left(\frac{\partial^2}{\partial x^2}-4\frac{\partial^2}{\partial x\partial y}+3\frac{\partial^2}{\partial y^2}\right)u + 2\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)u =$$

3.(25 pts.) Consider the partial differential equation

(*) $$u_{xx}-4u_{xy}+3u_{yy}+2u_x-2u_y=0.$$

(a) Classify the order and type (nonlinear, linear, homogeneous, inhomogeneous, elliptic, hyperbolic, parabolic) of (*).

(b) Find, if possible, the general solution of (*) in the xy-plane.

(a) (*) is a $\boxed{\text{second-order, linear, homogeneous}}$ p.d.e. 3

$B^2-4AC = (-4)^2 - 4(1)(3) = 4 > 0$. Therefore (*) is $\boxed{\text{hyperbolic}}$.

(b) $$u_{xx}-4u_{xy}+3u_{yy} = \left(\frac{\partial^2}{\partial x^2}-4\frac{\partial^2}{\partial x\partial y}+3\frac{\partial^2}{\partial y^2}\right)u = \left(\frac{\partial}{\partial x}-3\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)u$$

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Let $\begin{cases} \xi = 3x+y \\ \eta = x+y. \end{cases}$ Then $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial v}{\partial \eta}\dfrac{\partial \eta}{\partial x} = 3\dfrac{\partial v}{\partial \xi} + \dfrac{\partial v}{\partial \eta}$

and $\dfrac{\partial v}{\partial y} = \dfrac{\partial v}{\partial \xi}\dfrac{\partial \xi}{\partial y} + \dfrac{\partial v}{\partial \eta}\dfrac{\partial \eta}{\partial y} = \dfrac{\partial v}{\partial \xi} + \dfrac{\partial v}{\partial \eta}$.

That is, $\dfrac{\partial}{\partial x} = 3\dfrac{\partial}{\partial \xi} + \dfrac{\partial}{\partial \eta}$ and $\dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial \xi} + \dfrac{\partial}{\partial \eta}$. Therefore $\dfrac{\partial}{\partial x} - \dfrac{\partial}{\partial y} = 2\dfrac{\partial}{\partial \xi}$

and $\dfrac{\partial}{\partial x} - 3\dfrac{\partial}{\partial y} = -2\dfrac{\partial}{\partial \eta}$ so (*) is equivalent to

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$$\left(-2\frac{\partial}{\partial \eta}\right)\left(2\frac{\partial}{\partial \xi}\right)u + 2\left(2\frac{\partial}{\partial \xi}\right)u = 0 \quad \Rightarrow \quad \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) - \frac{\partial u}{\partial \xi} = 0.$$

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Replacing $\dfrac{\partial u}{\partial \xi}$ with w, we have $\dfrac{\partial w}{\partial \eta} - w = 0$. The solution to this

3

first-order p.d.e. is $w = c_1(\xi)e^{\eta}$. But $w = \dfrac{\partial u}{\partial \xi}$ so $u = \int c_1(\xi)e^{\eta}d\xi$

$= e^{\eta}\Big[\underbrace{\int c_1(\xi)d\xi}_{f(\xi)} + c_2(\eta)\Big]$. Therefore $\boxed{u(x,y) = f(3x+y)e^{x+y} + g(x+y)}$ 2

3

where f and g are C^2-functions of a single real variable.

4.(25 pts.) A long string with density $\rho = 2$ and tension $T = 8$ initially occupies the x-axis. The string is then plucked and begins to oscillate vertically. If the initial vertical displacement and vertical velocity of the string at position x are $\frac{1}{1+x^2}$ and $\sin(x)$, respectively, find the vertical displacement of the string as a function of the position x for all times $t > 0$. (Note: You may use appropriate formulas to solve this problem; you need not develop the solution "from scratch".)

$c^2 = 4 \Rightarrow c = 2$

5 $$\rho u_{tt} - T u_{xx} = 0 \Rightarrow 2u_{tt} - 8u_{xx} = 0 \Rightarrow u_{tt} - 4u_{xx} = 0.$$

4. $$u(x,0) = \frac{1}{1+x^2} = \varphi(x) \quad \text{and} \quad u_t(x,0) = \sin(x) = \psi(x).$$

By d'Alembert's formula

6 $$u(x,t) = \frac{1}{2}\left[\varphi(x-ct) + \varphi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)\,d\xi$$

2 $$= \frac{1}{2}\left[\frac{1}{1+(x-2t)^2} + \frac{1}{1+(x+2t)^2}\right] + \frac{1}{4}\int_{x-2t}^{x+2t} \sin(\xi)\,d\xi$$

3 $$= \frac{1}{2}\left[\frac{1}{1+(x-2t)^2} + \frac{1}{1+(x+2t)^2}\right] - \frac{1}{4}\cos(\xi)\Big|_{x-2t}^{x+2t}$$

1 $$= \frac{1}{2}\left[\frac{1}{1+(x-2t)^2} + \frac{1}{1+(x+2t)^2}\right] - \frac{1}{4}\Big[\overbrace{\cos(x+2t) - \cos(x-2t)}^{-2\sin(x)\sin(2t)}\Big]$$

$$\boxed{u(x,t) = \frac{1}{2}\left[\frac{1}{1+(x-2t)^2} + \frac{1}{1+(x+2t)^2}\right] + \frac{1}{2}\sin(x)\sin(2t)}$$

Bonus (25 pts.): A homogeneous body occupying the solid region

$$R = \{(x,y,z) \in \mathbf{R}^3 : 4 \le x^2 + y^2 + z^2 \le 100\}$$

is completely insulated and its initial temperature at position (x,y,z) in R is given by

$$\frac{50}{\sqrt{x^2+y^2+z^2}}.$$

(a) Write (without proof) the partial differential equation and initial/boundary conditions governing the temperature $u(x,y,z,t)$ at position (x,y,z) in R at time $t > 0$.

(b) Find the steady-state temperature that the body reaches after a long time.

12 pts.

(a)
$$\begin{cases} u_t - k\nabla^2 u = 0 & \text{if } 4 < x^2+y^2+z^2 < 100 \text{ and } t>0, \quad 4 \\ \dfrac{\partial u}{\partial n} = 0 & \text{if } x^2+y^2+z^2 = 4 \text{ or } x^2+y^2+z^2=100 \text{ and } t>0, \quad 4 \\ u(x,y,z,0) = \dfrac{50}{\sqrt{x^2+y^2+z^2}} & \text{if } 4 \le x^2+y^2+z^2 \le 100. \quad 4 \end{cases}$$

13 pts.

(b) The heat energy contained in R at time t is proportional to

$$E(t) = \iiint_R u(x,y,z,t)\,dV.$$

4

It is not hard to see that $E(t)$ is conserved since the body is completely insulated.
Therefore $E(t) = E(0) = \iiint_R \frac{50}{\sqrt{x^2+y^2+z^2}}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{10} \frac{50}{r} r^2 \sin\varphi \, dr\, d\varphi\, d\theta =$
$50\left(\frac{1}{2}r^2\right)\Big|_2^{10}\left(-\cos\varphi\right)\Big|_0^{\pi} 2\pi = 9600\pi$ for all $t>0$. But, denoting the (uniform) steady-state temperature reached by the body after long time by U,

$$E(0) = \lim_{t\to\infty} E(t) = \lim_{t\to\infty}\iiint_R u(x,y,z,t)\,dV = \iiint_R \lim_{t\to\infty} u(x,y,z,t)\,dV = \iiint_R U\,dV =$$

$U\,\mathrm{vol}(R)$. Thus

5

$$U = \frac{9600\pi}{\mathrm{vol}(R)} = \frac{9600\pi}{\frac{4}{3}\pi(10^3-2^3)} = \frac{3\cdot 9600\pi}{4\pi\cdot 992} = \boxed{\frac{225}{31}} \sim 7.258$$