1. (25 pts.) Consider a function of the form

\[ u(x,t) = A x^2 + B x t + C t^2 + D x + E t + F \]

where \( A, B, C, D, E, \) and \( F \) are constants.

(a) What is the most general form of \( u \) if it solves \( u_t - u_{xx} = 0 \) in the \( xt \)-plane? (Note: The correct answer will involve three arbitrary constants.)

(b) Find the solution of \( u_t - u_{xx} = 0 \) that satisfies \( u(x,0) = 3x^2 \) for all \(-\infty < x < \infty \).

(a) \[ 0 = u_t - u_{xx} = (Bx + 2Ct + E) - (2A) = Bx + 2Ct + (E - 2A) \]

Therefore \( B = 2C = E - 2A = 0 \), i.e. \( B = 0 \), \( C = 0 \), and \( E = 2A \). Thus

\[ u(x,t) = A x^2 + D x + 2 A t + F \]

(b) \[ 3x^2 = u(x,0) = A x^2 + D x + F \] so \( A = 3 \), \( D = F = 0 \).

Consequently

\[ u(x,t) = 3x^2 + 6t \]
2. (25 pts.) Consider the partial differential equation

\[ (y+1)u_x + 2xyu_y = 0. \]

(a) Find the characteristic curves of (*)

(b) Write the general solution of (*)

(c) Find the solution of (*) that satisfies the condition \( u(0, y) = ye^y \) for \( y > 0 \).

\[ \langle y+1, 2xy \rangle \cdot \nabla u = 0. \] Therefore \( u \) is constant along curves whose tangent line is parallel to \( \langle y+1, 2xy \rangle \) at a general point \((x,y)\) on the curve. I.e.

\[ \frac{dy}{dx} = \frac{2xy}{y+1} \Rightarrow \left( \frac{y+1}{y} \right) dy = 2x \, dx \quad \text{(Variables Separable)} \]

\[ \Rightarrow \int \left( 1 + \frac{1}{y} \right) dy = \int 2x \, dx \]

\[ \Rightarrow y + \ln(y) = x^2 + c \]

\[ y + \ln(y) - x^2 = c \]

\[ \text{Characteristic curves of (*)}. \]

(b) Along a characteristic curve \( u(x,y) = u(\sqrt{y + \ln(y) - c}, y) \)

\[ y = 1 \]

\[ u(x,y) = f(y + \ln(y) - x^2) \]

is the general solution of (*).

\[ f \text{ is a } C^1 \text{ function of a single real variable}. \]

(c) \[ ye^y = u(0, y) = f(y + \ln(y)) \] for all \( y > 0 \). But

\[ ye^y = e^{\ln(y)} \cdot e^y = e^{y + \ln(y)} \]

So \( e^{y + \ln(y)} = f(y + \ln(y)) \),

and hence \( f(z) = e^z \) for all real \( z \).

\[ u(x,y) = e^{y + \ln(y) - x^2} \]

\[ \text{This can also be written as } u(x,y) = ye^{y-x^2}. \]
3. (25 pts.) Consider the partial differential equation
\[ (*) \quad u_{xx} - 4u_{xy} + 3u_{yy} + 2u_x - 2u_y = 0. \]

(a) Classify the order and type (nonlinear, linear, homogeneous, inhomogeneous, elliptic, hyperbolic, parabolic) of (*)

(i) is a second-order, linear, homogeneous p.d.e.

\[ B^2 - 4AC = (-4)^2 - 4(1)(3) = 4 > 0. \] Therefore, (*) is \[ \text{hyperbolic}. \]

(b) \[ u_{xx} - 4u_{xy} + 3u_{yy} = \left( \frac{2}{\partial x} - 4 \frac{2}{\partial x \partial y} + 3 \frac{2}{\partial y} \right) u = \left( \frac{2}{\partial x} - 3 \frac{2}{\partial y} \right) \left( \frac{2}{\partial x} - 2 \right) u \]

Let \( \xi = 3x + y \) and \( \eta = x + y \).

Then \[ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} = 3 \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \]

and \[ \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \].

That is, \[ \frac{2}{\partial x} = 3 \frac{2}{\partial \xi} + \frac{2}{\partial \eta} \] and \[ \frac{2}{\partial y} = \frac{2}{\partial \xi} + \frac{2}{\partial \eta} \]. Therefore \[ \frac{2}{\partial x} - \frac{2}{\partial y} = \frac{2}{\partial \xi} \]

and \[ \frac{2}{\partial x} - 3 \frac{2}{\partial y} = -2 \frac{2}{\partial \eta} \] so (*) is equivalent to

\[ (-2 \frac{2}{\partial \eta})(\frac{2}{\partial \xi}) u + 2(\frac{2}{\partial \xi}) u = 0 \quad \Rightarrow \quad \frac{2}{\partial \eta}(\frac{2}{\partial \xi}) u - \frac{2}{\partial \xi} u = 0. \]

Replacing \( \frac{2}{\partial \xi} u \) with \( w \), we have \( \frac{2}{\partial \eta} w - w = 0 \). The solution to this first-order p.d.e. is \( w = c_1(e^{-2}) \). But \( w = \frac{2}{\partial \xi} u \) so \( u = \int c_1(e^{-2}) \eta \partial \xi \]

\[ = \int \left[ c_1(e^{-2}) \eta + c_2(\eta) \right] \partial \xi. \] Therefore \[ u(x,y) = f(3x+y) e^{x+y} + g(x+y) \]

where \( f \) and \( g \) are \( C^2 \)-functions of a single real variable.
4.(25 pts.) A long string with density $\rho = 2$ and tension $T = 8$ initially occupies the $x-$axis. The string is then plucked and begins to oscillate vertically. If the initial vertical displacement and vertical velocity of the string at position $x$ are $\frac{1}{1+x^2}$ and $\sin(x)$, respectively, find the vertical displacement of the string as a function of the position $x$ for all times $t > 0$. (Note: You may use appropriate formulas to solve this problem; you need not develop the solution “from scratch”.)

\[ c = 2 \]

By d'Alembert's formula

\[ u(x,t) = \frac{1}{2} \left[ \varphi(x-ct) + \varphi(x+ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(\xi) d\xi \]

\[ u(x,t) = \frac{1}{2} \left[ \frac{1}{1 + (x-2t)^2} + \frac{1}{1 + (x+2t)^2} \right] + \frac{1}{4} \int_{x-2t}^{x+2t} \sin(\xi) d\xi \]

\[ u(x,t) = \frac{1}{2} \left[ \frac{1}{1 + (x-2t)^2} + \frac{1}{1 + (x+2t)^2} \right] - \frac{1}{4} \cos(3) \]

\[ u(x,t) = \frac{1}{2} \left[ \frac{1}{1 + (x-2t)^2} + \frac{1}{1 + (x+2t)^2} \right] + \frac{1}{2} \sin(x) \sin(2t) \]

\[ u(x,t) = \frac{1}{2} \left[ \frac{1}{1 + (x-2t)^2} + \frac{1}{1 + (x+2t)^2} \right] + \frac{1}{2} \sin(x) \sin(2t) \]
Bonus (25 pts.): A homogeneous body occupying the solid region

\[ R = \{(x, y, z) \in \mathbb{R}^3 : 4 \leq x^2 + y^2 + z^2 \leq 100\} \]

is completely insulated and its initial temperature at position \((x, y, z)\) in \(R\) is given by

\[ u(x, y, z, 0) = \frac{50}{\sqrt{x^2 + y^2 + z^2}}. \]

(a) Write (without proof) the partial differential equation and initial/boundary conditions governing the temperature \(u(x, y, z, t)\) at position \((x, y, z)\) in \(R\) at time \(t > 0\).

(b) Find the steady-state temperature that the body reaches after a long time.

\[ u_t - k \nabla^2 u = 0 \quad \text{if} \quad 4 \leq x^2 + y^2 + z^2 < 100 \quad \text{and} \quad t > 0, \]

\[ \frac{\partial u}{\partial n} = 0 \quad \text{if} \quad x^2 + y^2 + z^2 = 4 \quad \text{or} \quad x^2 + y^2 + z^2 = 100 \quad \text{and} \quad t > 0, \]

\[ u(x, y, z, 0) = \frac{50}{\sqrt{x^2 + y^2 + z^2}} \quad \text{if} \quad 4 \leq x^2 + y^2 + z^2 \leq 100. \]

(b) The heat energy contained in \(R\) at time \(t\) is proportional to

\[ E(t) = \iiint_R u(x, y, z, t) \, dV. \]

It is not hard to see that \(E(t)\) is conserved since the body is completely insulated.

Therefore \(E(t) = E(0) = \iiint_R \frac{50}{\sqrt{x^2 + y^2 + z^2}} \, dV = \iint_0^{10} \frac{50}{r} r \, \sin \phi \, dr \, d\phi \, d\theta = 50 \left( \frac{1}{2} r^2 \right) \left[ \sin \phi \right]_0^\pi = 9600\pi \) for all \(t > 0\). But, denoting the

(uniform) steady-state temperature reached by the body after long time by \(U\),

\[ E(0) = \lim_{t \to 0} E(t) = \lim_{t \to 0} \iiint_R u(x, y, z, t) \, dV = \iiint_R \lim_{t \to 0} u(x, y, z, t) \, dV = \iiint_R U \, dV = \]

\[ U \cdot \text{Vol}(R). \]

Thus

\[ U = \frac{9600\pi}{\text{Vol}(R)} = \frac{9600\pi}{\frac{4}{3}\pi (10^3 - 2^3)} = \frac{3 \cdot 9600\pi}{4 \pi \cdot 992} = \frac{225}{31} \approx 7.258 \]