Mathematics 325 Exam II Fall 2000 Name: Dr. Grow

1. (30 pts.) Find a solution to

$$u_t - u_{xx} = 0 \quad \text{in} \quad -\infty < x < \infty,\ 0 < t < \infty,$$

satisfying

$$u(x,0) = e^{-x^2} \quad \text{for} \quad -\infty < x < \infty.$$

$$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} e^{-y^2}\, dy$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2-2xy+y^2}{4t}}}{\sqrt{4\pi t}} \cdot e^{-\frac{4ty^2}{4t}}\, dy$$

Complete square and "compensate".

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{(4t+1)y^2 - 2y\sqrt{4t+1}\left(\frac{x}{\sqrt{4t+1}}\right) + \frac{x^2}{4t+1}}{4t}} \cdot e^{\frac{-x^2 + \frac{x^2}{4t+1}}{4t}}}{\sqrt{4\pi t}}\, dy$$

$$= e^{\frac{-(4t+1)x^2 + x^2}{4t(4t+1)}} \int_{-\infty}^{\infty} \frac{e^{-\frac{\left(\sqrt{4t+1}\, y - \frac{x}{\sqrt{4t+1}}\right)^2}{4t}}}{\sqrt{4\pi t}}\, dy$$

Let $p = \dfrac{\sqrt{4t+1}\, y - \frac{x}{\sqrt{4t+1}}}{\sqrt{4t}}$

Then $dp = \sqrt{\dfrac{4t+1}{4t}}\, dy$

$$= \frac{e^{-\frac{x^2}{4t+1}}}{\sqrt{\pi}\cdot\sqrt{4t+1}} \int_{-\infty}^{\infty} e^{-p^2}\, dp$$

$$= \boxed{\frac{e^{-\frac{x^2}{4t+1}}}{\sqrt{4t+1}}}$$

(Valid for $t > -\frac{1}{4}$ and $-\infty < x < \infty$.)

2. (30 pts.) (a) Let $f = f(x)$ be an absolutely integrable function on $(-\infty, \infty)$, let c be a real constant, and define the translate of f by c according to the formula

$$f_c(x) = f(x-c) \qquad \text{for all } -\infty < x < \infty.$$

Show that the Fourier transform of f_c satisfies the relation

$$\mathcal{F}(f_c)(\xi) = e^{-i\xi c}\mathcal{F}(f)(\xi) \qquad \text{for all } -\infty < \xi < \infty.$$

(b) Use the Fourier transform method to solve

$$u_t - u_{xx} + u_x = 0 \qquad \text{in } -\infty < x < \infty,\ 0 < t < \infty,$$

subject to the initial condition

$$u(x,0) = \phi(x) \qquad \text{for all } -\infty < x < \infty.$$

(a) $$\mathcal{F}(f_c)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_c(x)e^{-i\xi x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-c)e^{-i\xi x}\,dx \leftarrow \left\{\begin{array}{l}\text{Let } y = x-c.\\ \text{Then } dy = dx\end{array}\right.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\,e^{-i\xi(y+c)}\,dy = e^{-i\xi c}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)e^{-i\xi y}\,dy = e^{-i\xi c}\mathcal{F}(f)(\xi).$$

(b) $$\mathcal{F}(u_t - u_{xx} + u_x)(\xi) = \mathcal{F}(0)(\xi)$$

$$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) - (i\xi)^2\mathcal{F}(u)(\xi) + i\xi\mathcal{F}(u)(\xi) = 0$$

$$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + (\xi^2+i\xi)\mathcal{F}(u)(\xi) = 0$$ ← Linear, first order de. An integrating factor is $\mu(t) = e^{\int(\xi^2+i\xi)\,dt} = e^{(\xi^2+i\xi)t}$.

$$\therefore\ e^{(\xi^2+i\xi)t}\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + (\xi^2+i\xi)e^{(\xi^2+i\xi)t}\mathcal{F}(u)(\xi) = 0$$

$$\frac{\partial}{\partial t}\left(e^{(\xi^2+i\xi)t}\mathcal{F}(u)(\xi)\right) = 0$$

$$e^{(\xi^2+i\xi)t}\mathcal{F}(u)(\xi) = c(\xi)$$

$$\mathcal{F}(u)(\xi) = c(\xi)e^{-\xi^2 t}\cdot e^{-i\xi t}$$

$$\therefore\ \mathcal{F}(\varphi)(\xi) = \mathcal{F}(u(\cdot,0))(\xi) = c(\xi)e^{-0}\cdot e^{i0} = c(\xi)$$

so $$\mathcal{F}(u)(\xi) = \mathcal{F}(\varphi)(\xi)\underbrace{e^{-\xi^2 t}}\cdot e^{-i\xi t}$$

Table entry I:

$$\mathcal{F}\left(e^{-a(\cdot)^2}\right)(\xi) = \frac{e^{-\xi^2/4a}}{\sqrt{2a}}.$$

Take $a = \frac{1}{4t}$ and rearrange:

$$\mathcal{F}\left(\frac{1}{\sqrt{2t}}e^{-(\cdot)^2/4t}\right)(\xi) = e^{-\xi^2 t}$$

Therefore $\quad \mathcal{F}(u)(\xi) = \underbrace{\mathcal{F}(\varphi)(\xi)\,\mathcal{F}\left(\frac{1}{\sqrt{2t}}e^{-\frac{(\cdot)^2}{4t}}\right)(\xi)}\, e^{-i\xi t}$

$\mathcal{F}(f*g)(\xi) = \sqrt{2\pi}\,\mathcal{F}(f)(\xi)\,\mathcal{F}(g)(\xi)$

$$= \frac{1}{\sqrt{2\pi}}\mathcal{F}\left(\varphi * \frac{1}{\sqrt{2t}}e^{-\frac{(\cdot)^2}{4t}}\right)(\xi)\, e^{-i\xi t}$$

$$= \mathcal{F}\left(\varphi * \frac{1}{\sqrt{4\pi t}}e^{-\frac{(\cdot)^2}{4t}}\right)(\xi)\, e^{-i\xi t}$$

Part (a) with $c = t$

$$= \mathcal{F}\left[\left(\varphi * \frac{1}{\sqrt{4\pi t}}e^{-\frac{(\cdot)^2}{4t}}\right)_t\right](\xi)$$

Therefore the inversion theorem implies

$$u(x,t) = \left(\varphi * \frac{1}{\sqrt{4\pi t}}e^{-\frac{(\cdot)^2}{4t}}\right)_t(x)$$

for all $-\infty < x < \infty$ and all $0 < t < \infty$.

I.e. $\quad u(x,t) = \left(\varphi * \frac{1}{\sqrt{4\pi t}}e^{-\frac{(\cdot)^2}{4t}}\right)(x-t)$

$$= \boxed{\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-t-y)^2}{4t}}}{\sqrt{4\pi t}}\,\varphi(y)\,dy}$$

for all $-\infty < x < \infty$ and all $0 < t < \infty$.

3. (40 pts.) (a) Find a solution to

(1) $$u_{tt} - u_{xx} + 2u_t = 0 \quad \text{in } 0 < x < \pi,\ 0 < t < \infty,$$

satisfying

(2) $$u(0,t) = 0 = u(\pi,t) \quad \text{for } t \ge 0,$$

(3) $$u(x,0) = 0 \quad \text{for } 0 \le x \le \pi,$$

(4) $$u_t(x,0) = \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x) \quad \text{for } 0 \le x \le \pi.$$

(b) If $u = u(x,t)$ satisfies (1)-(2), show that its energy

$$E(t) = \frac{1}{2}\int_0^{\pi} [u_t^2(x,t) + u_x^2(x,t)]dx$$

is nonincreasing on $0 \le t < \infty$.

(c) Is there only one solution to the problem in part (a)? Why or why not?

(a) (via separation of variables) We seek nontrivial solutions of (1)-(2)-(3) of the form $u(x,t) = X(x)T(t)$. Substituting in (1) gives $X(x)T''(t) - X''(x)T(t) + 2X(x)T'(t) = 0$, and hence $\dfrac{T''(t) + 2T'(t)}{T(t)} = \dfrac{X''(x)}{X(x)} = \text{constant} = -\lambda$. Substituting in (2)-(3) gives $X(0)T(t) = 0 = X(\pi)T(t)$ for $t \ge 0$ and $X(x)T(0) = 0$ for $0 \le x \le \pi$. Thus:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \text{ for } 0<x<\pi, \ X(0) = 0 = X(\pi); \\ T''(t) + 2T'(t) + \lambda T(t) = 0 \text{ for } t>0, \ T(0) = 0. \end{cases}$$

The eigenvalues are $\lambda_n = n^2$ $(n=1,2,3,\dots)$ and the corresponding eigenfunctions are $X_n(x) = \sin(nx)$ $(n=1,2,3,\dots)$. The functions of t satisfy $T_n''(t) + 2T_n'(t) + n^2T_n(t) = 0$. Then $T_n(t) = e^{\alpha_n t}$ leads to $\alpha_n^2 + 2\alpha_n + n^2 = 0$

$$\Rightarrow \alpha_n = \frac{-2 \pm \sqrt{4-4n^2}}{2} \quad (n=1,2,\dots)$$

$\therefore$ $\alpha_1 = -1$ and $\alpha_n = -1 \pm i\sqrt{n^2-1}$ for $n=2,3,\dots$

and $T_1(t) = a_1e^{-t} + b_1te^{-t}$, $T_n(t) = e^{-t}\left[a_n\cos(t\sqrt{n^2-1}) + b_n\sin(t\sqrt{n^2-1})\right]$ $(n \ge 2)$

The initial condition $T_n(0) = 0$ implies $a_1 = 0$ $(n=1)$ and $a_n = 0$ $(n \ge 2)$.

Thus (upto a constant factor) $u_1(x,t) = te^{-t}\sin(x)$ and

$u_n(x,t) = e^{-t}\sin(t\sqrt{n^2-1})\sin(nx)$ for $n \ge 2$. Consequently,

$$u(x,t) = b_1 t e^{-t} \sin(x) + \sum_{n=1}^{\infty} b_n e^{-t} \sin(t\sqrt{n^2-1}) \sin(nx)$$

is a (formal) solution of (1)-(2)-(3) for any choice of constants $b_1, b_2, \dots$.
Differentiating with respect to t (holding x fixed) yields

$$u_t(x,t) = b_1(1e^{-t} - te^{-t})\sin(x) + \sum_{n=1}^{\infty} b_n\left(\sqrt{n^2-1}\cos(t\sqrt{n^2-1}) - \sin(t\sqrt{n^2-1})\right)e^{-t}\sin(nx)$$

so we want to choose the constants $b_1, b_2, \dots$ so that, for all $0 \le x \le \pi$,

$$\tfrac{3}{4}\sin(x) - \tfrac{1}{4}\sin(3x) = u_t(x,0) = b_1 \sin(x) + \sum_{n=1}^{\infty} \sqrt{n^2-1}\, b_n \sin(nx).$$

By inspection, $b_1 = \frac{3}{4}$, $\sqrt{9-1}\, b_3 = -\frac{1}{4}$ (i.e. $b_3 = -\frac{1}{8\sqrt{2}}$), and all other $b_n = 0$ suffices. That is,

$$\boxed{u(x,t) = \tfrac{3}{4} t e^{-t} \sin(x) - \tfrac{1}{8\sqrt{2}} e^{-t} \sin(2t\sqrt{2}) \sin(3x)}$$

(b) $$E'(t) = \frac{d}{dt}\left\{ \frac{1}{2}\int_0^{\pi} [u_t^2(x,t) + u_x^2(x,t)]\,dx \right\} = \frac{1}{2}\int_0^{\pi} \frac{\partial}{\partial t}\left[u_t^2(x,t) + u_x^2(x,t)\right]dx$$

$$= \int_0^{\pi} [u_t(x,t) u_{tt}(x,t) + u_x(x,t) u_{xt}(x,t)]\,dx \overset{\text{by (1)}}{=} \int_0^{\pi} u_t(x,t)[u_{xx}(x,t) - 2u_t(x,t)]\,dx + \int_0^{\pi} \underbrace{u_x(x,t)}_{U}\, \underbrace{u_{xt}(x,t)\,dx}_{dV}$$

$$= \cancel{\int_0^{\pi} u_t(x,t) u_{xx}(x,t)\,dx} - 2\int_0^{\pi} u_t^2(x,t)\,dx + u_x(x,t)u_t(x,t)\Big|_{x=0}^{x=\pi} - \cancel{\int_0^{\pi} u_t(x,t) u_{xx}(x,t)\,dx}$$

$$= -2\int_0^{\pi} u_t^2(x,t)\,dx + u_x(\pi,t)\cancelto{0}{u_t(\pi,t)} - u_x(0,t)\cancelto{0}{u_t(0,t)}$$

$$u_t(0,t) = \lim_{k \to 0} \frac{\overbrace{u(0,t+k) - u(0,t)}^{0 \text{ by } (2)}}{k} = 0$$

and similarly for $u_t(\pi,t)$.

≤ 0. $\boxed{\text{Thus } E \downarrow \text{ on } 0 \le t < \infty.}$

(c) $\boxed{\text{There is only one solution to (1)-(2)-(3)-(4).}}$ To see this, suppose that $u = u(x,t)$ denotes the solution found above in part (a) and $u = u_1(x,t)$ is another solution. Then $v(x,t) = u(x,t) - u_1(x,t)$ solves

(1′) $v_{tt} - v_{xx} + 2v_t = 0$ in $0 < x < \pi$, $0 < t < \infty$,

(2′) $v(0,t) = 0 = v(\pi,t)$ for $t \ge 0$,

(OVER)

(3') $\quad v(x,0) = 0$ and (4') $v_t(x,0) = 0$ for $0 \le x \le \pi$.

By part (b), the energy function of v,

$$E(t) = \frac{1}{2}\int_0^{\pi} \left[v_t^2(x,t) + v_x^2(x,t) \right] dx,$$

is nonincreasing on $[0,\infty)$. Thus, for all $t \ge 0$,

$$0 \le E(t) \le E(0) = \frac{1}{2}\int_0^{\pi} \left[v_t^2(x,0) + v_x^2(x,0) \right] dx .$$

According to (3'), $v_x(x,0) = \lim\limits_{h\to 0} \dfrac{v(x+h,0) - v(x,0)}{h} = \lim\limits_{h\to 0} \dfrac{0-0}{h} = 0$ for all $0 < x < \pi$, and (4') says $v_t(x,0) = 0$ for $0 \le x \le \pi$; thus $E(0)=0$. It follows that $E(t)=0$ for all $t \ge 0$. The vanishing theorem then gives $v_t(x,t) = 0 = v_x(x,t)$ for all $0 \le x \le \pi$, $0 \le t$, from which it follows that $v(x,t) =$ constant. In light of (2') and (3'), $v(x,t) = 0$ for all $0 \le x \le \pi$ and all $t \ge 0$; i.e. $u_1 = u$.