Mathematics 325 Exam II Summer 2004 Name: Dr. Grow (1 pt.)

1.(33 pts.) Use Fourier transform methods to solve

$$u_t - u_{xx} + tu = 0 \quad \text{for} \quad -\infty < x < \infty,\ 0 < t < \infty,$$
$$u(x,0) = f(x) \quad \text{for} \quad -\infty < x < \infty.$$

$$\mathcal{F}(u_t - u_{xx} + tu)(\xi) = \mathcal{F}(0)(\xi)$$

4 $$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + \xi^2\mathcal{F}(u)(\xi) + t\mathcal{F}(u)(\xi) = 0$$

7 Integrating factor: $e^{\int(\xi^2+t)dt} = e^{\xi^2 t + \frac{t^2}{2}}$.

$$e^{\xi^2 t+\frac{t^2}{2}}\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + (\xi^2+t)e^{\xi^2 t+\frac{t^2}{2}}\mathcal{F}(u)(\xi) = 0$$

10 $$\frac{\partial}{\partial t}\left(e^{\xi^2 t+\frac{t^2}{2}}\mathcal{F}(u)(\xi)\right) = 0$$

$$e^{\xi^2 t+t^2/2}\mathcal{F}(u)(\xi) = A(\xi)$$

14 $$\mathcal{F}(u)(\xi) = A(\xi)e^{-\xi^2 t - t^2/2}$$

$$\hat{f}(\xi) = \mathcal{F}(u(\cdot,0))(\xi) = \mathcal{F}(u)(\xi)\Big|_{t=0} = A(\xi)$$

18 $$\therefore \mathcal{F}(u)(\xi) = \hat{f}(\xi)e^{-\xi^2 t}\cdot e^{-t^2/2}$$

Take $a = \frac{1}{4t}$ in Formula I in the table of Fourier transforms to get

$$\mathcal{F}\left(\frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{2t}}\right)(\xi) = e^{-\xi^2 t}.$$

22 $$\therefore \mathcal{F}(u)(\xi) = \mathcal{F}(f)(\xi)\cdot\mathcal{F}\left(\frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{2t}}\right)(\xi)\cdot e^{-t^2/2}$$

Using the property that

$$\mathcal{F}(g*h)(\xi) = \sqrt{2\pi}\,\mathcal{F}(g)(\xi)\mathcal{F}(h)(\xi)$$

gives

26 $$\mathcal{F}(u)(\xi) = \frac{1}{\sqrt{2\pi}}\mathcal{F}\left(f * \frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{2t}}\right)(\xi)\cdot e^{-t^2/2}$$

$$= \mathcal{F}\left(f * \frac{e^{-\frac{(\cdot)^2}{4t} - t^2/2}}{\sqrt{4\pi t}}\right)(\xi)$$

The inversion theorem then implies

30 $$u(x,t) = \left(f * \frac{e^{-\frac{(\cdot)^2}{4t} - t^2/2}}{\sqrt{4\pi t}}\right)(x)$$

$$= \int_{-\infty}^{\infty}\frac{e^{-\frac{(x-y)^2}{4t} - t^2/2}}{\sqrt{4\pi t}}f(y)\,dy$$

33 $$= \boxed{\frac{e^{-t^2/2}}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}e^{-\frac{(x-y)^2}{4t}}f(y)\,dy}$$

2.(33 pts.) (a) State (without providing a proof) the weak maximum/minimum principle for solutions to the diffusion equation.

(b) Show that solutions to the wave equation need not satisfy a maximum principle.

(c) Use the weak maximum/minimum principle to show that there is at most one solution to the Poisson equation with Dirichlet boundary conditions:

$$u_t - ku_{xx} = f(x,t) \quad \text{for} \quad 0 < x < L,\ 0 < t \leq T,$$
$$u(x,0) = \phi(x) \quad \text{for} \quad 0 \leq x \leq L,$$
$$u(0,t) = g(t) \quad \text{and} \quad u(L,t) = h(t) \quad \text{for} \quad 0 \leq t \leq T.$$

(a) Let $u = u(x,t)$ be a solution of $u_t - ku_{xx} = 0$ in $R = \{(x,t) : 0 < x < L,\ 0 < t \leq T\}$ and let u be continuous on $\overline{R} = \{(x,t) : 0 \leq x \leq L,\ 0 \leq t \leq T\}$. Then the maximum and minimum values of u on $\overline{R}$ are attained on $\overline{R} \setminus R$, i.e. either initially $(t=0)$ or on the lateral walls $(x=0$ or $x=L)$.

(b) $u(x,t) = \sin(\pi t)\sin(\pi x)$ solves the wave equation $u_{tt} - u_{xx} = 0$ in $R = \{(x,t) : 0 < x < 1,\ 0 < t < 1\}$ and is continuous on $\overline{R} = \{(x,t) : 0 \leq x \leq 1,\ 0 \leq t \leq 1\}$. Yet $u = 0$ on $\overline{R} \setminus R$ while the maximum value of u on $\overline{R}$ is $u(\frac{1}{2}, \frac{1}{2}) = 1$.

(c) Suppose $u = u_1(x,t)$ and $u = u_2(x,t)$ solve the problem in part (c) and are continuous functions on $\overline{R} = \{(x,t) : 0 \leq x \leq L,\ 0 \leq t \leq T\}$. Then $w(x,t) = u_1(x,t) - u_2(x,t)$ solves $u_t - ku_{xx} = 0$ in $R = \{(x,t) : 0 < x < L,\ 0 < t \leq T\}$ and is continuous on $\overline{R}$. Therefore the maximum and minimum of w on $\overline{R}$ are attained on $\overline{R} \setminus R$. However $w(x,0) = 0$ for $0 \leq x \leq L$ and $w(0,t) = 0 = w(L,t)$ for $0 \leq t \leq T$. Therefore $w = 0$ on $\overline{R} \setminus R$, and it follows that $w(x,t) = 0$ for all (x,t) in $\overline{R}$; i.e. $u_1(x,t) = u_2(x,t)$ for all (x,t) in $\overline{R}$.

3.(33 pts.) Use energy methods to show that there is at most one solution to the Poisson equation with Neumann boundary conditions:

$$u_t - ku_{xx} = f(x,t) \quad \text{for} \quad 0 < x < L,\ 0 < t < \infty,$$
$$u(x,0) = \phi(x) \quad \text{for} \quad 0 \le x \le L,$$
$$u_x(0,t) = g(t) \quad \text{and} \quad u_x(L,t) = h(t) \quad \text{for} \quad 0 \le t < \infty.$$

Suppose $u = u_1(x,t)$ and $u = u_2(x,t)$ solve the above problem. Then $w(x,t) =$
5 $u_1(x,t) - u_2(x,t)$ solves

① $u_t - ku_{xx} = 0$ for $0<x<L,\ 0<t<\infty$,

② $u(x,0) = 0$ for $0 \le x \le L$,

9 ③-④ $u_x(0,t) = 0 = u_x(L,t)$ for $0 \le t < \infty$.

14 Consider the energy function of w: $E(t) = \int_0^L w^2(x,t)\,dx$. Then $\frac{dE}{dt} =$

19 $\int_0^L \frac{\partial}{\partial t}\left(w^2(x,t)\right)dx = \int_0^L 2w(x,t)w_t(x,t)\,dx \overset{①}{=} 2k\int_0^L \overbrace{w(x,t)}^{v}\overbrace{w_{xx}(x,t)\,dx}^{dv} =$

24 $2k\cdot w(x,t)w_x(x,t)\Big|_{x=0}^{L} - 2k\int_0^L w_x^2(x,t)\,dx \overset{③-④}{=} 0 - 2k\int_0^L w_x^2(x,t)\,dx \le 0.$

Therefore E is a nonincreasing function of t on $[0,\infty)$. Therefore, for $0<t<\infty$,

Clear!

$0 \le E(t) \le E(0) = \int_0^L w^2(x,0)\,dx \overset{②}{=} \int_0^L 0\,dx = 0$. It follows that

29 $\int_0^\infty w^2(x,t)\,dx = E(t) = 0$ for all $0 \le t < \infty$. Consequently, the vanishing theorem implies $w^2(x,t) = 0$ for all $-\infty < x < \infty$ for each $t \ge 0$, and

33 hence $w(x,t) = 0$ for all $-\infty<x<\infty$ and $0 \le t < \infty$. That is, $u_1(x,t) = u_2(x,t)$ in the upper halfplane.

Bonus.(33 pts.) (a) Use Fourier transform methods to show that, under appropriate hypotheses, a solution to the Poisson initial value problem:

$$u_t - ku_{xx} = f(x,t) \quad \text{for } -\infty < x < \infty,\ 0 < t < \infty,$$
$$u(x,0) = \phi(x) \quad \text{for } -\infty < x < \infty,$$

is given by

$$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4kt}}}{\sqrt{4k\pi t}}\phi(y)dy + \int_0^t \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4k(t-\tau)}}}{\sqrt{4k\pi(t-\tau)}} f(y,\tau)dyd\tau .$$

(b) Use the formula in part (a) to solve the Poisson initial value problem with $\phi(x) \equiv 0$ and

$$f(x,t) = \begin{cases} x & \text{if } |x| < t, \\ 0 & \text{otherwise.} \end{cases}$$

(a) $\mathcal{F}(u_t - ku_{xx})(\xi) = \mathcal{F}(f(x,t))(\xi) \equiv F(\xi,t)$

4 $\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + k\xi^2\mathcal{F}(u)(\xi) = F(\xi,t)$. An integrating factor is $e^{\int k\xi^2 dt} = e^{k\xi^2 t}$.

8 $$\underbrace{e^{k\xi^2 t}\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + k\xi^2 e^{k\xi^2 t}\mathcal{F}(u)(\xi)}_{\frac{\partial}{\partial t}\left(e^{k\xi^2 t}\mathcal{F}(u)(\xi)\right)} = e^{k\xi^2 t}F(\xi,t) \Rightarrow e^{k\xi^2 t}\mathcal{F}(u)(\xi) = A(\xi) + \int_0^t e^{k\xi^2\tau}F(\xi,\tau)\,d\tau$$

12 $$\therefore \mathcal{F}(u)(\xi) = A(\xi)e^{-k\xi^2 t} + e^{-k\xi^2 t}\int_0^t e^{k\xi^2\tau}F(\xi,\tau)\,d\tau$$

$$= A(\xi)e^{-k\xi^2 t} + \int_0^t e^{-k\xi^2(t-\tau)}F(\xi,\tau)\,d\tau$$

$$\mathcal{F}(\varphi)(\xi) = \mathcal{F}(u(\cdot,0))(\xi) = \mathcal{F}(u)(\xi)\Big|_{t=0} = A(\xi).$$

16 $$\therefore \mathcal{F}(u)(\xi) = \mathcal{F}(\varphi)(\xi)e^{-k\xi^2 t} + \int_0^t e^{-k\xi^2(t-\tau)}F(\xi,\tau)\,d\tau$$

From formula I in the table of Fourier transforms, $\mathcal{F}\left(\frac{e^{-\frac{(\cdot)^2}{4kt}}}{\sqrt{2kt}}\right)(\xi) = e^{-k\xi^2 t}$.

(OVER)

$$\therefore\quad \mathcal{F}(u)(\xi) = \mathcal{F}(\varphi)(\xi)\,\mathcal{F}\left(\frac{e^{-\frac{(\cdot)^2}{4kt}}}{\sqrt{2kt}}\right)(\xi) + \int_0^t \mathcal{F}\left(\frac{e^{-\frac{(\cdot)^2}{4k(t-\tau)}}}{\sqrt{2k(t-\tau)}}\right)(\xi)\,\mathcal{F}\big(f(\cdot,\tau)\big)d\tau$$

Using the convolution formula: $\mathcal{F}(g*h)(\xi) = \sqrt{2\pi}\,\mathcal{F}(g)(\xi)\mathcal{F}(h)(\xi)$ *yields*

$$\mathcal{F}(u)(\xi) = \frac{1}{\sqrt{2\pi}}\mathcal{F}\left(\varphi * \frac{e^{-\frac{(\cdot)^2}{4kt}}}{\sqrt{2kt}}\right)(\xi) + \frac{1}{\sqrt{2\pi}}\int_0^t \mathcal{F}\left(f(\cdot,\tau) * \frac{e^{-\frac{(\cdot)^2}{4k(t-\tau)}}}{\sqrt{2k(t-\tau)}}\right)(\xi)\,d\tau$$

$$= \mathcal{F}\left(\varphi * \frac{e^{-\frac{(\cdot)^2}{4kt}}}{\sqrt{4k\pi t}} + \int_0^t f(\cdot,\tau) * \frac{e^{-\frac{(\cdot)^2}{4k(t-\tau)}}}{\sqrt{4k\pi(t-\tau)}}\,d\tau\right)(\xi).$$

By the inversion theorem,

$$u(x,t) = \left(\varphi * \frac{e^{-\frac{(\cdot)^2}{4kt}}}{\sqrt{4k\pi t}}\right)(x) + \int_0^t \left(f(\cdot,\tau) * \frac{e^{-\frac{(\cdot)^2}{4k(t-\tau)}}}{\sqrt{4k\pi(t-\tau)}}\right)(x)\,d\tau$$

$$= \boxed{\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4kt}}}{\sqrt{4k\pi t}}\varphi(y)\,dy + \int_0^t\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4k(t-\tau)}}}{\sqrt{4k\pi(t-\tau)}} f(y,\tau)\,dy\,d\tau}.$$

(b) $$u(x,t) = \int_0^t\int_{-\tau}^{\tau} \frac{e^{-\frac{(x-y)^2}{4k(t-\tau)}}}{\sqrt{4k\pi(t-\tau)}}\,y\,dy\,d\tau$$

Let $p = \dfrac{y-x}{\sqrt{4k(t-\tau)}}$.

Then $dp = \dfrac{dy}{\sqrt{4k(t-\tau)}}$

$$= \int_0^t \frac{1}{\sqrt{\pi}}\int_{\frac{-\tau-x}{\sqrt{4k(t-\tau)}}}^{\frac{\tau-x}{\sqrt{4k(t-\tau)}}} e^{-p^2}\left(\sqrt{4k(t-\tau)}\,p + x\right)dp\,d\tau$$

(Cont.)

$$u(x,t) = \int_0^t \frac{\sqrt{4k(t-\tau)}}{(-2)\sqrt{\pi}} \int_{\frac{-\tau-x}{\sqrt{4k(t-\tau)}}}^{\frac{\tau-x}{\sqrt{4k(t-\tau)}}} e^{-p^2}(-2p)\,dp\,d\tau + \int_0^t \frac{x}{\sqrt{\pi}} \int_{\frac{-\tau-x}{\sqrt{4k(t-\tau)}}}^{\frac{\tau-x}{\sqrt{4k(t-\tau)}}} e^{-p^2}\,dp\,d\tau$$

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$$= \boxed{\begin{aligned} &\int_0^t \frac{\sqrt{4k(t-\tau)}}{2\sqrt{\pi}} \left[-e^{-\frac{(\tau-x)^2}{4k(t-\tau)}} + e^{\frac{(\tau+x)^2}{4k(t-\tau)}} \right] d\tau \\ &\qquad + \frac{x}{2} \int_0^t \left[\operatorname{Erf}\left(\frac{\tau-x}{\sqrt{4k(t-\tau)}} \right) + \operatorname{Erf}\left(\frac{\tau+x}{\sqrt{4k(t-\tau)}} \right) \right] d\tau \end{aligned}}$$

A Brief Table of Fourier Transforms

$f(x)$	$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$
A. $\begin{cases} 1 & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b\xi)}{\xi}$
B. $\begin{cases} 1 & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{-ic\xi} - e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C. $\frac{1}{x^2 + a^2}$ $(a > 0)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\|\xi\|}}{a}$
D. $\begin{cases} x & \text{if } 0 < x \le b, \\ 2b - x & \text{if } b < x < 2b, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{-1 + 2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E. $\begin{cases} e^{-ax} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$ $(a > 0)$	$\frac{1}{(a + i\xi)\sqrt{2\pi}}$
F. $\begin{cases} e^{ax} & \text{if } b < x < c, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{(a-i\xi)c} - e^{(a-i\xi)b}}{(a - i\xi)\sqrt{2\pi}}$
G. $\begin{cases} e^{iax} & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b(\xi-a))}{\xi - a}$
H. $\begin{cases} e^{iax} & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{i}{\sqrt{2\pi}} \frac{e^{ic(a-\xi)} - e^{id(a-\xi)}}{a - \xi}$
I. e^{-ax^2} $(a > 0)$	$\frac{1}{\sqrt{2a}} e^{-\xi^2/(4a)}$
J. $\frac{\sin(ax)}{x}$ $(a > 0)$	$\begin{cases} 0 & \text{if } \|\xi\| \ge a, \\ \sqrt{\pi/2} & \text{if } \|\xi\| < a. \end{cases}$