Mathematics 325   Exam II   Name: Dr. Grow
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1.(25 pts.) Find a solution to

$$u_t - u_{xx} = 0 \quad \text{for } -\infty < x < \infty \text{ and } 0 < t < \infty,$$

which satisfies $u(x,0) = x^3$ for $-\infty < x < \infty$. You may find the following identities useful:

$$\int_{-\infty}^{\infty} e^{-p^2}\,dp = \sqrt{\pi}, \qquad \int_{-\infty}^{\infty} p e^{-p^2}\,dp = 0, \qquad \int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp = \frac{\sqrt{\pi}}{2}, \qquad \int_{-\infty}^{\infty} p^3 e^{-p^2}\,dp = 0.$$

Candidate for a solution: $u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}}\varphi(y)\,dy = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} y^3\,dy$.

(Since $\varphi(y) = y^3$ is not bounded on $(-\infty,\infty)$, the theorems do not guarantee this is a solution so we will need to check our answer at the end.) Make the substitution $p = \frac{y-x}{\sqrt{4t}}$ in this integral.

$$u(x,t) = \frac{1}{\cancel{\sqrt{4\pi t}}}\int_{-\infty}^{\infty} e^{-p^2}\left(\sqrt{4t}\,p + x\right)^3 dp\,\cancel{\sqrt{4t}}$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}\left((4t)^{3/2}p^3 + 3(4t)p^2x + 3(4t)^{1/2}px^2 + x^3\right)dp$$

$$= \frac{(4t)^{3/2}}{\sqrt{\pi}}\underbrace{\int_{-\infty}^{\infty} e^{-p^2}p^3\,dp}_{0} + \frac{12tx}{\sqrt{\pi}}\underbrace{\int_{-\infty}^{\infty} e^{-p^2}p^2\,dp}_{\frac{\sqrt{\pi}}{2}} + \frac{3(4t)^{1/2}x^2}{\sqrt{\pi}}\underbrace{\int_{-\infty}^{\infty} e^{-p^2}p\,dp}_{0} + \frac{x^3}{\sqrt{\pi}}\underbrace{\int_{-\infty}^{\infty} e^{-p^2}\,dp}_{\sqrt{\pi}}$$

$$\boxed{u(x,t) = 6tx + x^3}$$

Check: $u_t - u_{xx} = 6x - 6x \stackrel{\checkmark}{=} 0$

$u(x,0) \stackrel{\checkmark}{=} x^3$

2.(25 pts.) Use Fourier transform methods to find a formula for the solution to

$$u_t - u_{xx} + 2tu = 0 \quad \text{for } -\infty < x < \infty \text{ and } 0 < t < \infty,$$

subject to the initial condition $u(x,0) = \phi(x)$ for $-\infty < x < \infty$.

$$\mathcal{F}(u_t - u_{xx} + 2tu)(\xi) = \mathcal{F}(0)(\xi) = 0$$

$$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) - (i\xi)^2\mathcal{F}(u)(\xi) + 2t\mathcal{F}(u)(\xi) = 0$$

$$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + (\xi^2 + 2t)\mathcal{F}(u)(\xi) = 0$$

Linear 1st order ODE in the variable t; ξ is just a parameter.

Integrating factor: $\mu(t) = e^{\int(\xi^2+2t)dt} = e^{\xi^2 t + t^2}$.

$$e^{\xi^2 t + t^2}\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + (\xi^2+2t)e^{\xi^2 t+t^2}\mathcal{F}(u)(\xi) = 0$$

$$\frac{\partial}{\partial t}\left(e^{\xi^2 t+t^2}\mathcal{F}(u)(\xi)\right) = 0 \quad \Rightarrow \quad e^{\xi^2 t+t^2}\mathcal{F}(u)(\xi) = c(\xi)$$

$\therefore \mathcal{F}(u)(\xi) = c(\xi)e^{-\xi^2 t - t^2}$. Applying the initial condition yields

$$\mathcal{F}(\varphi)(\xi) = \mathcal{F}(u(\cdot,0))(\xi) = c(\xi)e^{0} = c(\xi).$$

$$\mathcal{F}(u)(\xi) = \mathcal{F}(\varphi)(\xi)\,e^{-\xi^2 t}\cdot e^{-t^2}$$

$$= \mathcal{F}(\varphi)(\xi)\,\mathcal{F}\left(\frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{2t}}\right)(\xi)\,e^{-t^2}$$

$$= \frac{1}{\sqrt{2\pi}}\mathcal{F}\left(\varphi * \frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{2t}}\right)(\xi)\cdot e^{-t^2}$$

$$= \mathcal{F}\left(e^{-t^2}\varphi * \frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{4\pi t}}\right)(\xi)$$

$$\therefore u(x,t) = e^{-t^2}\varphi * \frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{4\pi t}}(x) = \boxed{\frac{e^{-t^2}}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}}\varphi(y)\,dy}.$$

Table Entry I:

$$\mathcal{F}\left(\sqrt{2a}\,e^{-a(\cdot)^2}\right)(\xi) = e^{-\frac{\xi^2}{4a}}.$$

Take $t = \frac{1}{4a}$ to get

$$\mathcal{F}\left(\frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{2t}}\right)(\xi) = e^{-\xi^2 t}$$

$$\mathcal{F}(f*g)(\xi) = \sqrt{2\pi}\,\mathcal{F}(f)(\xi)\,\mathcal{F}(g)(\xi)$$

3. (25 pts.) Find a solution to

$$① \quad u_{tt} - u_{xx} = 0 \quad \text{for } 0 < x < 1,\ 0 < t < \infty,$$

ubject to

$$②\text{-}③ \quad u_x(0,t) = u_x(1,t) = 0 \quad \text{for } t \geq 0,$$

and

$$u(x,0) \overset{④}{=} \cos^2(\pi x), \quad u_t(x,0) \overset{⑤}{=} 0 \quad \text{for } 0 \leq x \leq 1.$$

Bonus. (10 pts.) Show that there is only one solution to this boundary value problem.

$u(x,t) = X(x)T(t)$ in ①-②-③-⑤ leads to

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$$\begin{cases} X''(x) + \lambda X(x) = 0, & X'(0) = 0 = X'(1), \\ T''(t) + \lambda T(t) = 0, & T'(0) = 0. \end{cases}$$

The eigenvalues/eigenfunctions are $\left.\begin{aligned} \lambda_n &= (n\pi)^2 \\ X_n(x) &= \cos(n\pi x) \end{aligned}\right\}$ $n = 0, 1, 2, \ldots$

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The corresponding solution to the T-equation/T-B.C. when $\lambda = \lambda_n = (n\pi)^2$ is (up to a constant factor) $T_n(t) = \cos(n\pi t)$. Thus a formal solution to ①-②-③-⑤ is

18 $u(x,t) = \sum_{n=0}^{\infty} a_n X_n(x) T_n(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)\cos(n\pi t)$. In order to satisfy ④

20 we require $\cos^2(\pi x) = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$ for $0 \leq x \leq 1$. Applying the identity $\cos^2(A) = \frac{1+\cos(2A)}{2}$ this becomes

$$\frac{1}{2} + \frac{1}{2}\cos(2\pi x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \quad \text{for } 0 \leq x \leq 1.$$

25 Choose $\frac{1}{2} = a_0 = a_2$ and all other $a_n = 0$. Solution: $\boxed{u(x,t) = \frac{1}{2} + \frac{1}{2}\cos(2\pi x)\cos(2\pi t)}$.

Bonus: Let $u = v(x,t)$ be another solution to ①-②-③-④-⑤ and consider $w(x,t) = u(x,t) - v(x,t)$ and its energy function

$$E(t) = \int_0^1 \left[\tfrac{1}{2} w_t^2(x,t) + \tfrac{1}{2} w_x^2(x,t) \right] dx \qquad (t \geq 0).$$

Note that $w = w(x,t)$ solves

(OVER)

$$\begin{cases} w_{tt} - w_{xx} \overset{⑥}{=} 0 & \text{for } 0<x<1,\ 0<t<\infty, \\ w_x(1,t) \overset{⑦}{=} 0 \overset{⑧}{=} w_x(0,t) & \text{for } t \geq 0, \\ w(x,0) \overset{⑨}{=} 0 \overset{⑩}{=} w_t(x,0) & \text{for } 0 \leq x \leq 1. \end{cases}$$

We claim that $E(t) = \text{constant}$ for all $t \geq 0$. To see this observe that for $t \geq 0$,

$$\frac{dE}{dt} = \int_0^1 \frac{\partial}{\partial t}\left[\tfrac{1}{2} w_t^2(x,t) + \tfrac{1}{2} w_x^2(x,t)\right] dx$$

$$= \int_0^1 \left[w_t(x,t) w_{tt}(x,t) + w_x(x,t) w_{xt}(x,t)\right] dx$$

$$= \int_0^1 \left[w_t(x,t) w_{xx}(x,t) + w_x(x,t) w_{xt}(x,t)\right] dx \qquad (\text{using } ⑥)$$

$$= \int_0^1 \frac{\partial}{\partial x}\left[w_t(x,t) w_x(x,t)\right] dx$$

$$= w_t(x,t) w_x(x,t) \Big|_{x=0}^{1}$$

$$= w_t(1,t) w_x(1,t) - w_t(0,t) w_x(0,t).$$

$$= 0 \qquad (\text{by } ⑦ \text{ and } ⑧).$$

Therefore $E(t) = E(0)$ for all $t \geq 0$. Equation ⑨ implies $w_x(x,0) = \lim_{h \to 0} \frac{w(x+h,0) - w(x,0)}{h} = 0$, and thus $E(0) = \int_0^1 \left[\tfrac{1}{2} w_t^2(x,0) + \tfrac{1}{2} w_x^2(x,0)\right] dx = 0$. Since $E(t) = 0$ for all $t \geq 0$, the vanishing theorem implies $w_t(x,t) = w_x(x,t) = 0$ for all $0 \leq x \leq 1$ and all $0 \leq t < \infty$. Consequently $w(x,t) = \text{constant}$ on $0 \leq x \leq 1,\ 0 \leq t < \infty$. But ⑨ shows that this constant must be zero. That is, $u(x,t) = v(x,t)$ for all $0 \leq x \leq 1,\ 0 \leq t < \infty$. This shows that there is only one solution to ①-②-③-④-⑤.

4. (25 pts.) (a) Give a clear mathematical statement (no proof is required) of the weak maximum principle for solutions to the heat equation.

(b) State in your own words the physical meaning of this principle for temperature distributions in a rod.

(c) By exhibiting an appropriate partial differential equation and a solution to this equation in an appropriate region, show that solutions to the wave equation need not satisfy a maximum principle.

(a) Let $u = u(x,t)$ be a solution to $u_t - ku_{xx} = 0$ in $R: 0 < x < \ell,\ 0 < t \le T$ such that u is continuous on $\overline{R}: 0 \le x \le \ell,\ 0 \le t \le T$. Then the maximum value of u on $\overline{R}$ occurs when $t = 0$ or when $x = 0$ or ℓ.

8 (b) Consider a thin rod of length ℓ which is insulated on the lateral surface. Then the hottest temperature in the rod occurs either initially (i.e. when $t=0$) or at one of the ends (i.e. $x=0$ or $x=\ell$) of the rod.

9 (c) Consider the solution $u(x,t) = \sin(\pi x)\sin(\pi t)$ of the wave equation $u_{tt} - u_{xx} = 0$ in the square $S: 0 \le x \le 1, 0 \le t \le 1$. Note that the maximum value of u on S is attained at $(\frac{1}{2}, \frac{1}{2})$ $[\ u(\frac{1}{2},\frac{1}{2}) = 1\ ]$, an interior point of S, while $u = 0$ at all points of the boundary of S. Thus solutions to the wave equation need not satisfy a maximum principle.