1. (35 pts.) (a) Show that the operator \( T = -\frac{d^2}{dx^2} \) is hermitian on \( V = \{ f \in C^2[0,1]: f''(0) = 0 = f(1) \} \) equipped with the standard inner product \( \langle f, g \rangle = \int_0^1 f(x)g(x)dx \).

(b) Find all the eigenvalues and corresponding eigenfunctions of \( T \) on \( V \).

(c) Does the set of functions \( \left\{ \cos \left( \left( n + \frac{1}{2} \right) \pi x \right) \right\}_{n=0}^{\infty} \) form an orthogonal system on \([0,1]\) with the standard inner product? Justify your answer.

(d) Show that the Fourier series of \( f(x) = 1 - x^2 \) with respect to \( \left\{ \cos \left( \left( n + \frac{1}{2} \right) \pi x \right) \right\}_{n=0}^{\infty} \) on \([0,1]\) is

\[
\sum_{n=0}^{\infty} \frac{32(-1)^n \cos \left( \left( n + \frac{1}{2} \right) \pi x \right)}{\pi^3 (2n+1)^3}.
\]

(e) Write the partial sum consisting of the first two terms of the above Fourier series for \( f \). On the same coordinate axes, sketch the graph of this partial sum and the graph of \( f \).

(f) Assume that for every \( x \) in \([0,1]\), \( f(x) = 1 - x^2 \) is equal to its Fourier series in part (d). Find the sum of \( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \).

6. (a) Let \( f \) and \( g \) belong to \( V \). Then, using the same coordinate axes, sketch the graph of this partial sum and the graph of \( f \).

\[
\langle Tf, g \rangle = \int_0^1 -f''(x)g(x)dx = \left[ f(x)g'(x) - f'(x)g(x) \right]_0^1 = \int_0^1 f(x)g'(x)dx.
\]

But \( f(0) = 0 = g(1) \) and \( g'(0) = 0 = f'(1) \) so \( \langle Tf, g \rangle = \langle f, Tg \rangle \); that is, \( T \) is hermitian on \( V \).

6. (b) Since \( T = -\frac{d^2}{dx^2} \) is hermitian on \( V \), all its eigenvalues are real numbers. In fact, since \( -f(x)f'(x) \bigg|_{x=0}^1 = 0 \) for all real-valued functions \( f \) in \( V \), all the eigenvalues of \( T \) on \( V \) are positive, say \( \lambda = \beta^2 \). Then \( Tf = \lambda f \) on \( V \) becomes \( f''(x) + \beta^2 f(x) = 0 \), \( f(0) = 0 \), \( f(1) = 0 \). Then
The function \( f(x) = A \cos(\beta x) + B \sin(\beta x) \) and \( f'(x) = -\beta A \sin(\beta x) + \beta B \cos(\beta x) \).  
\[ 0 = f'(0) = \beta B \]
\[ \Rightarrow B = 0. \]
\[ 0 = f(1) = A \cos(\beta) \Rightarrow \beta = \beta_n = \left( \frac{2n+1}{2} \right) \pi = (n+\frac{1}{2}) \pi \quad (n=0,1,2,\ldots). \]
Therefore the eigenvalues and eigenfunctions are \( \lambda_n = (n+\frac{1}{2})^2 \pi^2 \) and \( \psi_n(x) = \cos((n+\frac{1}{2})x) \).

16 (c) Yes, \( \{ \cos((n+\frac{1}{2})\pi x) \}_{n=0}^{\infty} \) is an orthogonal system on \([0,1]\) because they are eigenfunctions of a hermitian operator corresponding to the distinct eigenvalues \( \lambda_n = (n+\frac{1}{2})^2 \pi^2 \quad (n=0,1,2,\ldots) \).

16 (d) \[
1 - x^2 = \sum_{n=0}^{\infty} c_n \cos((n+\frac{1}{2})\pi x) \quad \text{where} \quad c_n = \frac{\langle 1-x^2, \cos((n+\frac{1}{2})\pi x) \rangle}{\langle \cos((n+\frac{1}{2})\pi x), \cos((n+\frac{1}{2})\pi x) \rangle} (n=0,1,2,\ldots)
\]
\[
\langle \cos((n+\frac{1}{2})\pi x), \cos((n+\frac{1}{2})\pi x) \rangle = \int_0^1 \cos((n+\frac{1}{2})\pi x) dx = \int_0^1 \left[ \frac{1}{2} + \frac{1}{2} \cos(2(n+\frac{1}{2})\pi x) \right] dx = \left[ \frac{x + \frac{1}{2}(\cos(2n+1)\pi)}{2(n+1)\pi} \right]_0^1 = \frac{1}{2}. \]
\[ \langle 1-x^2, \cos((n+\frac{1}{2})\pi x) \rangle = \int_0^1 (1-x^2) \cos((n+\frac{1}{2})\pi x) dx = \int_0^1 \left( \frac{1-x^2}{2} \sin((n+\frac{1}{2})\pi x) \right) dx = \left. \left[ \frac{1}{(n+\frac{1}{2})\pi} \sin((n+\frac{1}{2})\pi x) \right] \right|_0^1 = \frac{2(-1)^n}{(n+\frac{1}{2})\pi} \frac{\sin((n+\frac{1}{2})\pi)}{(n+\frac{1}{2})\pi^3} = \frac{2(-1)^n}{(n+\frac{1}{2})\pi^3} \]  \[ c_n = 2 \cdot \frac{2 \cdot (-1)^n}{(n+\frac{1}{2})\pi^3} \]

16 (e) \[ y = 1 - x^2 \]

The graphs of \( y = 1-x^2 \) and \( \frac{y}{\pi^3} \left[ \cos((n+\frac{1}{2})\pi x) - \frac{1}{2} \cos(\pi x) \right] \)

are nearly indistinguishable on \([0,1]\).

16 (f) \[ 1 - x^2 = \sum_{n=0}^{\infty} \frac{32(-1)^n \cos((n+\frac{1}{2})\pi x)}{\pi^3 (2n+1)^3} \]
for all \( 0 \leq x \leq 1 \). Let \( x = 0 \)

to get \[ 1 = \sum_{n=0}^{\infty} \frac{32(-1)^n}{\pi^3 (2n+1)^3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}. \]
2. (35 pts.) Find a solution to

\[ u_t - u_{xx} = 0 \quad \text{if} \quad 0 < x < 1, \ 0 < t < \infty, \]

which satisfies

\[ u_t(0, t) = u_t(1, t) = 0 \quad \text{if} \quad t \geq 0, \]

and

\[ u(x, 0) = 1 - x^2, \ u_t(x, 0) = 0 \quad \text{if} \quad 0 \leq x \leq 1. \]

(Hint: You may find the results of problem 1 useful.)

Bonus (15 pts.): Show that the solution to the problem above is unique.

We use separation of variables. We seek nontrivial solutions to

**\[ u(x, t) = \Psi(x)T(t), \]**

substituting this form into

\[ \begin{cases} 
\Psi''(x) + \lambda \Psi(x) = 0, & \Psi(0) = \Psi(1) = 0, \\
T''(t) + \lambda T(t) = 0, & T'(0) = 0.
\end{cases} 
\]

By #1, the eigenvalues and eigenfunctions are \( \lambda_n = (n + \frac{1}{2}) \pi^2 \) and \( \Psi_n(x) = \cos((n + \frac{1}{2}) \pi x) \) for \( n = 0, 1, 2, ... \). The solution to the \( t \)-problem corresponding to \( \lambda = \lambda_n \) is (up to a constant factor) \( T_n(t) = \cos((n + \frac{1}{2}) \pi t) \). Therefore

\[ u(x, t) = \sum_{n=0}^{\infty} c_n \cos((n + \frac{1}{2}) \pi x) \cos((n + \frac{1}{2}) \pi t) \]

is a formal solution to **\[ 1 - x^2 = u(x, 0) = \sum_{n=0}^{\infty} c_n \cos((n + \frac{1}{2}) \pi x) \quad \text{for all} \ 0 \leq x \leq 1. \]**

By #1, \( c_n = \frac{32(-1)^n}{\pi^3 (2n+1)^3} \) for \( n = 0, 1, 2, ... \). Thus the solution to **\[ u(x, t) = \sum_{n=0}^{\infty} \frac{32(-1)^n \cos((n + \frac{1}{2}) \pi x) \cos((n + \frac{1}{2}) \pi t)}{\pi^3 (2n+1)^3} \].**

**Bonus:** We use energy methods to show that the solution is unique.

Suppose there were another solution \( v(x, t) \) to the problem. Then
$W(x,t) = u(x,t) - n(x,t)$ would solve the problem $\theta - \sigma - \zeta - \eta$ and

$5'$ \quad u(x,0) = 0 \quad \text{if} \quad 0 \leq x \leq 1.$

Consider the energy function of $w$:

$$E(t) = \int_0^1 \left[ \frac{1}{2} w_t^2(x,t) + \frac{1}{2} w_x^2(x,t) \right] dx.$$ 

Then

$$\frac{dE}{dt} = \int_0^1 \frac{\partial}{\partial t} \left[ \frac{1}{2} w_t^2(x,t) + \frac{1}{2} w_x^2(x,t) \right] dx$$

$$= \int_0^1 \left[ w_t(x,t) w_{tt}(x,t) + w_x(x,t) w_{xt}(x,t) \right] dx$$

As

$$= \int_0^1 \left[ w_t(x,t) w_{xx}(x,t) + w_x(x,t) w_{xt}(x,t) \right] dx$$

$$= \int_0^1 \frac{\partial}{\partial x} \left[ w_t(x,t) w_x(x,t) \right] dx$$

$$= w_t(x,t) w_x(x,t) \bigg|_{x=0}^1$$

But $5'$ and $\theta$ yield $w_t(1,t) = 0$ and $w_x(0,t) = 0$, so $\frac{dE}{dt} = 0$.

Thus $E$ is constant:

$$E(t) = E(0) = \int_0^1 \left[ \frac{1}{2} w_t^2(x,0) + \frac{1}{2} w_x^2(x,0) \right] dx = 0$$

by $\eta$ and $5'$ for all $t > 0$. By the vanishing theorem, $w_t(x,t) = 0$ and $w_x(x,t) = 0$ for all $0 \leq x \leq 1$ and each fixed $t > 0$. Therefore

$5'$ implies $w(x,t) = 0$; i.e., $n(x,t) = u(x,t)$ and the solution obtained above is unique.
Use the method of separation of variables to find a solution of the beam equation
\[ u_{xx} + u_{xxxx} = 0 \] if \( 0 < x < 1, \ 0 < t < \infty, \)
which satisfies the boundary conditions
\[ u(0, t) = u(1, t) = u_x(0, t) = u_x(1, t) = 0 \] if \( t \geq 0, \)
and the initial conditions
\[ u(x, 0) = 2\sin(\pi x) - 3\sin(5\pi x) \] and \( u_t(x, 0) = 0 \) if \( 0 \leq x \leq 1. \)

We seek nontrivial solutions to \( 0 - 2 - 3 - 4 - 5 - 6 \) of the form \( u(x, t) = \overline{X}(x)T(t). \)

Substituting into the PDE and the BC/ICs \( 2 - 6 \) leads to

\[
\begin{align*}
\frac{\overline{X}''(x)}{\overline{X}(x)} - \lambda \overline{X}(x) &= 0, \\
\overline{X}(0) &= \overline{X}(1) = \overline{X}''(0) = \overline{X}''(1) = 0,
\end{align*}
\]

It is easy to check that the operator \( \frac{d^4}{dx^4} \) is hermitian on \( V = \{ f \in C^4[0,1] : f(0) = f''(0) = f''(1) = f''(1) = 0 \} \), so the eigenvalues of the problem are real. In fact, if \( \lambda \) is an eigenvalue let \( 0 \neq \overline{X} \in V \) such that \( \overline{X}''(x) = \lambda \overline{X} \). Then two integrations by parts shows that

\[
\lambda \langle \overline{X}, \overline{X} \rangle = \langle \lambda \overline{X}, \overline{X} \rangle = \langle \overline{X}'', \overline{X} \rangle = \int_0^1 \overline{X}(x) \overline{X}(x) dx = \left( \frac{\overline{X}(0)}{\overline{X}'(0)} - \frac{\overline{X}(0)}{\overline{X}'(0)} \right) + \int_0^1 \overline{X}''(x) \overline{X}(x) dx = 0.
\]

But \( \overline{X}(0) = \overline{X}'(0) = 0 \) and \( \overline{X}(1) = \overline{X}'(1) = 0 \) so \( \lambda \langle \overline{X}, \overline{X} \rangle = \langle \overline{X}'', \overline{X}'' \rangle > 0. \)

Since \( \langle \overline{X}, \overline{X} \rangle > 0 \) it follows that \( \lambda > 0. \)

**Case \( \lambda > 0 \):** Let \( \lambda = \kappa^2 \) where \( \kappa > 0. \) The eigenvalue equation becomes \( \overline{X}''(x) - \kappa^2 \overline{X}(x) = 0. \)

\( \overline{X}(x) = e^{\kappa x} \) leads to \( e^{\kappa x} - e^{\kappa x} = 0 \Rightarrow e^{\kappa x} = 0 \Rightarrow r^4 - \kappa^4 = 0 \Rightarrow (r^2 + \kappa^2)(r^2 - \kappa^2) = 0 \)

\( \Rightarrow r = \pm \kappa, \pm i \kappa. \) Thus \( \overline{X}(x) = c_1 e^{\kappa x} + c_2 e^{-\kappa x} + c_3 e^{i \kappa x} + c_4 e^{-i \kappa x} \) is the general solution of the DE. Equivalently, \( \overline{X}(x) = c_1 \cosh(\kappa x) + c_2 \sinh(\kappa x) + c_3 \cos(\kappa x) + c_4 \sin(\kappa x) \)

and hence \( \overline{X}''(x) = \kappa^2 c_1 \cosh(\kappa x) + \kappa^2 c_2 \sinh(\kappa x) - \kappa^2 c_3 \cos(\kappa x) - \kappa^2 c_4 \sin(\kappa x). \)

0 = \overline{X}(0) = c_1 + c_3 \quad \text{and} \quad 0 = \overline{X}''(0) = \kappa^2 c_1 - \kappa^2 c_3 \quad \text{implies} \quad c_1 = c_3 = 0. \quad \text{Then} \quad c_1 = c_3 = 0. \quad \text{Then} \quad c_1 = c_3 = 0. \quad \text{Then} \quad 0 = \overline{X}(1) = c_2 \sinh(\kappa x) + c_4 \sin(\kappa x) \quad \text{and} \quad 0 = \overline{X}''(1) = \kappa^2 c_2 \sinh(\kappa x) - \kappa^2 c_4 \sin(\kappa x). \)
Adding these last two equations yields \( 0 = 2\pi c \sin(\pi x) \Rightarrow c_2 = 0 \).

Then \( 0 = x c_4 \sin(\pi x) \) so \( \sin(\pi x) = 0 \) is the eigenvalue condition. Therefore \( \lambda_n = \pi n \) and \( \pi_n \sin(\pi n x) \) are the eigenvalues and eigenfunctions, respectively, where \( n = 1, 2, 3, \ldots \).

Case \( \lambda = 0 \): The eigenvalue equation becomes \( \pi^2 \pi_n = 0 \) so \( \pi_n = c_1 x + c_2 x + c_3 x + c_4 \) and \( \pi^2 \pi_n = 6 c_1 x + 2 c_2 \). Then \( 0 = \pi(0) = c_4 \) and \( 0 = \pi(0) = 2 c_2 \) \( \Rightarrow c_2 = 0 \).

Also \( 0 = \pi(1) = c_1 + c_3 \) and \( 0 = \pi(1) = 6c_1 \) \( \Rightarrow c_1 = 0 = c_3 \). Therefore, there is no nontrivial solution so zero is not an eigenvalue.

The solution of the t-equation \( T_n(t) + \lambda_n T_n(t) = 0 \) \( \Rightarrow T_n(t) + (\pi_n^2)^4 T_n(t) = 0 \)

is \( T_n(t) = c_1 \cos(\pi n \pi t) + c_2 \sin(\pi n \pi t) \). Hence \( T_n(t) = -\pi^2 c_1 \sin(\pi n \pi t) + \pi^2 c_2 \cos(\pi n \pi t) \).

So \( 0 = T_n(0) = \pi^2 c_2 \Rightarrow c_2 = 0 \). Thus \( T_n(t) = \cos(\pi n \pi t) \), up to a constant factor.

By the superposition principle, \( u(x, t) = \sum_{n=1}^{N} c_n \sin(\pi n x) \cos(\pi n \pi t) \) solves the homogeneous portion of the problem \( 1-2-3-4-5-6 \) for every integer \( N \geq 1 \) and all constants \( c_1, \ldots, c_N \). We want to satisfy the inhomogeneous condition \( \int 2 \sin(\pi x) - 3 \sin(5\pi x) = u(x, 0) = \sum_{n=1}^{N} c_n \sin(\pi n x) \) for all \( 0 \leq x \leq 1 \).

Consequently we may take \( N = 5 \) and \( c_1 = 2, c_5 = -3, \) and other \( c_n = 0 \). That is,

\[
U(x, t) = 2 \sin(\pi x) \cos(\pi^2 \pi t) - 3 \sin(5\pi x) \cos(25\pi^2 t)
\]

is a solution of the problem \( 1-2-3-4-5-6-7 \).

Note: Using the energy function \( E(t) = \int_0^1 \left[ \frac{1}{2} u_x^2(x, t) + \frac{1}{2} u_{xx}^2(x, t) \right] dx \), it can be shown that this solution is unique.


Math 325
Exam III
Summer 2006

n = 15

mean = 52.7

standard deviation = 20.6

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