Mathematics 325 — Exam IV — Fall 2005 — Name: Dr. Grow (1 pt.)

1.(33 pts.) (a) Show that the Fourier sine series of $f(x)=x$ on the unit interval $0\le x\le 1$ is

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(n\pi x)}{n}.$$

(b) For which values of x in $[0,1]$ does the Fourier sine series of $f(x)=x$ converge pointwise to $f(x)$? Justify your answer.

(c) Does the Fourier sine series of $f(x)=x$ converge in the mean square sense on $[0,1]$? Why?

(d) Does the Fourier sine series of $f(x)=x$ converge uniformly on $[0,1]$? Why?

(e) Apply Parseval's identity to find the sum of $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

6 (a) $$b_n = \frac{\langle f, \sin(n\pi\cdot)\rangle}{\langle \sin(n\pi\cdot), \sin(n\pi\cdot)\rangle} = 2\int_0^1 \underbrace{x}_{u}\,\underbrace{\sin(n\pi x)\,dx}_{dv} = 2\left(\left.-\frac{x\cos(n\pi x)}{n\pi}\right|_0^1 - \int_0^1 \cancel{-\frac{\cos(n\pi x)}{n\pi}\,dx}^{\,0}\right)$$

$$b_n = \frac{-2\cos(n\pi)}{n\pi} = \frac{2(-1)^{n+1}}{n\pi}. \quad\therefore\quad \boxed{x \sim \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}\sin(n\pi x)}{\pi n}}$$

7 (b) $f(x)=x$ is continuous on $[0,1]$; $f'(x)=1$ is continuous on $[0,1]$ } Theorem 4(i) implies that the Fourier sine series of f converges to $f(x)=x$ for all x in $(0,1)$.

End points: If $x=0$ then $\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}\sin(n\pi 0)}{\pi n} = 0 = f(0)$.

If $x=1$ then $\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}\sin(n\pi)}{\pi n} = 0 \ne 1 = f(1)$.

$\boxed{\text{The Fourier sine series of } f(x)=x \text{ converges to } f(x)=x \text{ for all } x \text{ in } [0,1).}$

6 (c) $\boxed{\text{Yes,}}$ by Theorem 3 the Fourier sine series converges to $f(x)=x$ in the mean-square sense because $\int_0^1 |f(x)|^2dx = \int_0^1 x^2dx = \frac{1}{3} < \infty$.

7 (d) $\boxed{\text{No}}$, the Fourier sine series does not converge to $f(x)=x$ uniformly on $[0,1]$ because the Fourier sine series of $f(x)=x$ does not converge pointwise to $f(x)=x$ at $x=1$.

7 (e) $$\sum_{n=1}^{\infty}|A_n|^2\int_0^1|X_n(x)|^2dx = \int_0^1|f(x)|^2dx \Rightarrow \sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{\pi n}\right)^2\cdot\frac{1}{2} = \frac{1}{3} \Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}.$$

$$\overbrace{u_{xx}+u_{yy}+u_{zz}=x^2+y^2+z^2}^{\nabla^2 u = r^2}$$

2.(33 pts.) Solve $u_{xx}+u_{yy}+u_{zz}=x^2+y^2+z^2$ in the spherical shell $\overbrace{1<x^2+y^2+z^2<4}^{1<r<2}$ with u vanishing on both the inner and outer boundaries.

Due to rotational invariance of the p.d.e., the region, and the b.c.'s, we assume that $u=u(r,\theta,\varphi)=u(r)$, independent of the angles θ and φ. Thus

8 pts to here

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\,\cancel{\frac{\partial u}{\partial\varphi}}^{\,0}\right)+\frac{1}{r^2\sin^2\varphi}\cancel{\frac{\partial^2 u}{\partial\theta^2}}^{\,0}=r^2$$

16 pts. to here.

reduces to the o.d.e. $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)=r^2 \Rightarrow \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)=r^4 \Rightarrow r^2\frac{\partial u}{\partial r}=\frac{r^5}{5}+c_1$

$$\Rightarrow \frac{\partial u}{\partial r}=\frac{r^3}{5}+c_1r^{-2} \Rightarrow u=\frac{r^4}{20}-\frac{c_1}{r}+c_2$$ 24 pts. to here.

$$\left.\begin{aligned} 0=u(1)&=\frac{1}{20}-c_1+c_2\\ 0=u(2)&=\frac{16}{20}-\frac{c_1}{2}+c_2\end{aligned}\right\}\Rightarrow 0=\frac{15}{20}+\frac{c_1}{2}\Rightarrow c_1=-\frac{3}{2}\Rightarrow c_2=-\frac{31}{20}$$

28 pts. to here.

30 pts. to here.

$$\therefore \boxed{u(r)=\frac{r^4}{20}+\frac{3}{2r}-\frac{31}{20}}$$ 33 pts. to here.

In rectangular cartesian coordinates the solution is

$$\boxed{u(x,y,z)=\frac{(x^2+y^2+z^2)^2}{20}+\frac{3}{2\sqrt{x^2+y^2+z^2}}-\frac{31}{20}}$$

Note: A maximum principle argument shows that the above solution is the unique solution to the Poisson equation with Dirichlet boundary conditions.

3.(33 pts.) Solve $u_{xx}+u_{yy}+u_{zz}=0$ in the unit cube $0<x<1,\ 0<y<1,\ 0<z<1,$ given that $u(x,y,1)=x\sin(2\pi y)$ for $0\le x\le 1,\ 0\le y\le 1,$ and that u satisfies homogeneous Dirichlet boundary conditions on the other five faces of the cube.

$u(x,y,z) = X(x)Y(y)Z(z)$ in the homogeneous portion of the above problem leads to

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = \lambda \quad \text{and} \quad -\frac{Y''(y)}{Y(y)} = \frac{Z''(z)}{Z(z)} - \lambda = \mu, \quad \text{and the b.c.'s}$$

18 pts to here.

$X(0)=0=X(1),\ Y(0)=0=Y(1),\ Z(0)=0$. Thus

9 pts. to here.

$$\begin{cases} X''(x)+\lambda X(x)=0, \quad X(0)=0=X(1) & \Rightarrow \quad \lambda_\ell=(\ell\pi)^2, \quad X_\ell(x)=\sin(\ell\pi x) \quad (\ell=1, \\ Y''(y)+\mu Y(y)=0, \quad Y(0)=0=Y(1) & \Rightarrow \quad \mu_m=(m\pi)^2, \quad Y_m(y)=\sin(m\pi y) \quad (m=1, \\ Z''(z)-(\lambda+\mu)Z(z)=0, \quad Z(0)=0 & \Rightarrow \quad Z_{\ell,m}(z)=\sinh\left(\pi z\sqrt{\ell^2+m^2}\right) \end{cases}$$

27 pts. to here.

$$\therefore\ u(x,y,z) = \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty} c_{\ell,m}\sin(\ell\pi x)\sin(m\pi y)\sinh\left(\pi z\sqrt{\ell^2+m^2}\right) \quad \text{is a formal solution}$$

to the homogeneous part of the problem. We want to choose the constants $c_{\ell,m}$ in such a way that $x\sin(2\pi y) = u(x,y,1) = \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty} c_{\ell,m}\sinh(\pi\sqrt{\ell^2+m^2})\sin(\ell\pi x)\sin(m\pi y)$

for all $0\le x\le 1,\ 0\le y\le 1$. By inspection, we must have $c_{\ell,2}\sinh(\pi\sqrt{\ell^2+4}) = b_\ell = \ell^{th}$ sine coefficient of $f(x)=x$, and all other $c_{\ell,m}=0$. By problem 1(a), $c_{\ell,2} = \dfrac{b_\ell}{\sinh(\pi\sqrt{\ell^2+4})} = \dfrac{2(-1)^{\ell+1}}{\pi\ell\sinh(\pi\sqrt{\ell^2+4})}$.

Therefore

$$\boxed{u(x,y,z) = \sin(2\pi y)\sum_{\ell=1}^{\infty}\frac{2(-1)^{\ell+1}\sin(\ell\pi x)\sinh\left(\pi z\sqrt{\ell^2+4}\right)}{\pi\ell\sinh\left(\pi\sqrt{\ell^2+4}\right)}}$$

33 pts to here.

Note: A maximum principle argument shows that the solution above is the unique solution to $\nabla^2 u=0$ in the unit cube satisfying the Dirichlet boundary conditions.

Bonus (33 pts.): Let $P = P(x,y,z)$ be a polynomial with real coefficients in the real variables x, y, z such that

(1) $\nabla^2 P = 0$ for all (x,y,z) in $\mathbf{R}^3$, and

(2) the polynomial $x^2+y^2+z^2$ divides $P(x,y,z)$.

Show that $P(x,y,z)=0$ for all (x,y,z) in $\mathbf{R}^3$.

(General form of a polynomial in three variables.)

Write $P(x,y,z) = \sum_{i,j,k} c_{i,j,k} x^i y^j z^k$ where the sum is over a finite number of nonnegative integers i, j, and k, and where each coefficient $c_{i,j,k}$ is a real number. By hypothesis (2),

$$(*) \qquad P(x,y,z) = (x^2+y^2+z^2)\,Q(x,y,z) \qquad \text{for all } x, y, \text{ and } z$$

where $Q = Q(x,y,z)$ is a polynomial with real coefficients. Consider the differential operator $P\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \sum_{i,j,k} c_{i,j,k}\left(\frac{\partial}{\partial x}\right)^i\left(\frac{\partial}{\partial y}\right)^j\left(\frac{\partial}{\partial z}\right)^k$ and similarly for $Q\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. Then using $(*)$ and (1) yields

$$(**) \quad P\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)P(x,y,z) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)Q\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)P(x,y,z)$$

$$= Q\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2}\right)$$

$$= 0 \qquad \text{for all } x, y, \text{ and } z.$$

Observe that $\left(\frac{\partial}{\partial x}\right)^\ell x^i = \begin{cases} \frac{i!}{(i-\ell)!}x^{i-\ell} & \text{if } 0 \le \ell \le i, \\ 0 & \text{if } i < \ell, \end{cases}$

so $$\left(\frac{\partial}{\partial x}\right)^\ell\left(\frac{\partial}{\partial y}\right)^m\left(\frac{\partial}{\partial z}\right)^n x^i y^j z^k = \begin{cases} \frac{i!}{(i-\ell)!}\frac{j!}{(j-m)!}\frac{k!}{(k-n)!}x^{i-\ell}y^{j-m}z^{k-n} & \text{if } 0\le \ell\le i,\ 0\le m\le j \text{ and } 0\le n\le k, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$(***) \quad \left.\left(\frac{\partial}{\partial x}\right)^\ell\left(\frac{\partial}{\partial y}\right)^m\left(\frac{\partial}{\partial z}\right)^n x^i y^j z^k\right|_{(x,y,z)=(0,0,0)} = \begin{cases} i!\,j!\,k! & \text{if } (\ell,m,n)=(i,j,k), \\ 0 & \text{otherwise.} \end{cases}$$

Combining (**) and (***), we see that

$$\sum_{i,j,k} i!\,j!\,k!\,c_{i,j,k}^{2} = P\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) P(x,y,z)\Bigg|_{(x,y,z)=(0,0,0)} = 0 .$$

Thus $c_{i,j,k} = 0$ for all i, j, and k, and hence $P(x,y,z) = 0$ for all x, y, and z.