Mathematics 415 | Final Exam | Name: Dr. Grow
Fall 2007

This is an open-book, open-notes test. That is, while solving the problems on this examination you may refer at any time to your textbook, Royden's *Real Analysis*, or to the lecture notes you have taken for Math 415 this semester. The total number of points on this examination is 300.

1.(50 pts.) Let f be an absolutely continuous, 1-periodic function on the real line such that $f' \in L^2[0,1]$. Show that:

(a) the Fourier transforms of f' and f are related by $\hat{f'}(n) = 2\pi in\hat{f}(n)$ for all integers n;

(b) f has an absolutely convergent Fourier series: $\sum_{n=-\infty}^{\infty}\left|\hat{f}(n)e^{2\pi int}\right| < \infty$.

2.(50 pts.) Let E be a Lebesgue measurable subset of the real line. Show that

$$\lim_{\varepsilon\to 0^+}\frac{m\left(E\bigcap(x-\varepsilon,x+\varepsilon)\right)}{2\varepsilon}$$

is 1 a.e. on E and 0 a.e. on the complement of E.

3.(50 pts.) Let m denote Lebesgue measure on $(0,\infty)$. For any Lebesgue measurable subset E of $(0,\infty)$, define

$$\mu_1(E) = \sum_{n=1}^{\infty}\frac{1}{n^3}\int_{E\bigcap[n,n+1)} x\,dm,$$

$$\mu_2(E) = \int_{E\bigcap[1,\infty)}\frac{1}{x^2}\,dm.$$

Is m absolutely continuous with respect to μ_2? Is μ_2 absolutely continuous with respect to μ_1? Explain why or why not, and find the corresponding Radon-Nikodym derivatives, if they exist.

Problems 4 through 6 are interconnected and refer to the group $\mathbf{T}$ consisting of the points in the interval $[0,1)$ with "wrap around" addition. That is, the sum of x and y is the usual sum $x+y$ of two real numbers in the interval $[0,1)$ if $x+y$ is less than 1, and the sum of x and y is $x+y-1$ otherwise. (For those of you who are familiar with group theory, $\mathbf{T}$ is the additive quotient group $\mathbf{R}/\mathbf{Z}$.) The Lebesgue measure of subsets of $\mathbf{T}$ is the usual Lebesgue measure of the real line restricted to the interval $[0,1)$. When working any of the problems 4 through 6, you may assume the truth of the results from any preceding problem. (For example, this will allow you to solve problem 6, even if you could not successfully solve problems 4 and 5.)

4.(50 pts.) If E and F are subsets of $\mathbf{T}$, define $E+F=\{t+\tau : t\in E,\ \tau\in F\}$. The sum of any finite number of sets is defined similarly. A set E is called a basis for $\mathbf{T}$ if there exists a positive integer N such that $E+E+\ldots+E$ (N times) is equal to $\mathbf{T}$. Show that every subset E of $\mathbf{T}$ with positive Lebesgue measure is a basis. (Hint: If E has positive Lebesgue measure then $E+E$ contains an interval.)

5.(50 pts.) A character χ of $\mathbf{T}$ is a homomorphism from $\mathbf{T}$ into $\mathbf{C}^*$, the multiplicative group of nonzero complex numbers. In other words, χ is function from $\mathbf{T}$ into the nonzero complex numbers

$\boxed{\#1.}$ Let f be absolutely continuous and 1-periodic on the real line with $f' \in L^2[0,1]$. Then integration-by-parts implies

(a) $$\widehat{f'}(n) = \int_0^1 f'(t)e^{-2\pi i n t}\,dt$$

$$= f(1) - f(0) + 2\pi i n\int_0^1 f(t)e^{-2\pi i n t}\,dt$$

$$= 2\pi i n \hat{f}(n)$$

for every integer n. Therefore, by Cauchy-Schwarz and Parseval,

(b) $$\sum_{n=-\infty}^{\infty} \left|\hat{f}(n)e^{2\pi i n t}\right| = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|$$

$$= |\hat{f}(0)| + \sum_{n=1}^{\infty}\left(\left|\frac{\widehat{f'}(n)}{2\pi n}\right| + \left|\frac{\widehat{f'}(-n)}{-2\pi n}\right|\right)$$

$$\leq |\hat{f}(0)| + \frac{1}{2\pi}\left(\sum_{n=-\infty}^{\infty} |\widehat{f'}(n)|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}\frac{2}{n^2}\right)^{1/2}$$

$$= |\hat{f}(0)| + \frac{1}{2\sqrt{3}}\|f'\|_{L^2} < \infty.$$

$\boxed{\#2.}$ Let E be a Lebesgue measurable subset of the real line and define

$$F(x) = \int_0^x \chi_E(t)\,dm \qquad (x \in \mathbb{R}).$$

Then Lebesgue's differentiation theorem (cf. Theorem 5.10) implies $F'(x) = \chi_E(x)$ for a.e. x in $\mathbb{R}$. That is,

$$\begin{aligned}
\lim_{\varepsilon \to 0} \frac{F(x+\varepsilon) - F(x-\varepsilon)}{2\varepsilon} &= \lim_{\varepsilon \to 0} \left(\frac{F(x+\varepsilon) - F(x)}{2\varepsilon} \right) + \lim_{\varepsilon \to 0} \left(\frac{F(x-\varepsilon) - F(x)}{-2\varepsilon} \right) \\
&= \frac{1}{2} F'(x) + \frac{1}{2} F'(x) \\
&= \chi_E(x)
\end{aligned}$$

for a.e. x in $\mathbb{R}$. In other words,

$$\begin{aligned}
\lim_{\varepsilon \to 0^+} \frac{m\left(E \cap (x+\varepsilon, x-\varepsilon)\right)}{2\varepsilon} &= \lim_{\varepsilon \to 0^+} \frac{\int_{x-\varepsilon}^{x+\varepsilon} \chi_E(t)\,dm}{2\varepsilon} \\
&= \lim_{\varepsilon \to 0^+} \frac{F(x+\varepsilon) - F(x-\varepsilon)}{2\varepsilon} \\
&= \chi_E(x) \\
&= \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}
\end{aligned}$$

for a.e. x in $\mathbb{R}$.

$\boxed{\#3.}$ m is <u>not</u> absolutely continuous with respect to μ_2 because

10 $$\mu_2\big((0,1)\big) = \int_{(0,1)\cap[1,\infty)} \frac{1}{x^2}\,dm = \int_{\phi} \frac{1}{x^2}\,dm = 0 \qquad \text{and} \quad m\big((0,1)\big) = 1.$$

μ_2 <u>is</u> absolutely continuous with respect to μ_1 because

10 $$\mu_2(E) = \int_{E\cap[1,\infty)} \frac{1}{x^2}\,dm = \sum_{n=1}^{\infty} \int_{E\cap[n,n+1)} \frac{1}{x^2}\,dm \le \sum_{n=1}^{\infty} \int_{E\cap[n,n+1)} \frac{x}{n^3}\,dm = \mu_1(E)$$

for all measurable sets $E \subseteq (0,\infty)$. Therefore $\mu_1(E)=0$ clearly implies $\mu_2(E)=0$.

Note that for all measurable subsets E of $(0,\infty)$,

$$\mu_1(E) = \sum_{n=1}^{\infty} \frac{1}{n^3}\int_{E\cap[n,n+1)} x\,dm = \sum_{n=1}^{\infty} \frac{1}{n^3}\int_E x\,\chi_{[n,n+1)}(x)\,dm = \int_E \sum_{n=1}^{\infty} \frac{x\,\chi_{[n,n+1)}(x)}{n^3}\,dm$$

10 so $\mu_1 << m$ and $\left[\dfrac{d\mu_1}{dm}\right] = \displaystyle\sum_{n=1}^{\infty} \frac{x}{n^3}\chi_{[n,n+1)}(x)$ $\quad m$-a.e. in $(0,\infty)$.

Therefore $\mu_2 << \mu_1 << m$ so $\left[\dfrac{d\mu_2}{dm}\right] = \left[\dfrac{d\mu_2}{d\mu_1}\right]\left[\dfrac{d\mu_1}{dm}\right]$ (see #11.34(c)).

10 From the definition of μ_2 we see that $\left[\dfrac{d\mu_2}{dm}\right] = \dfrac{1}{x^2}\chi_{[1,\infty)}(x)$ $\quad m$-a.e.

10 Therefore
$$\left[\frac{d\mu_2}{d\mu_1}\right] = \frac{\left[\frac{d\mu_2}{dm}\right]}{\left[\frac{d\mu_1}{dm}\right]} = \begin{cases} 0 & \text{if } 0 < x < 1, \\[2ex] \dfrac{1/x^2}{\sum_{n=1}^{\infty} \frac{x}{n^3}\chi_{[n,n+1)}(x)} & \text{if } 1 \le x < \infty, \end{cases}$$

(cont.)

$$\left[\frac{d\mu_2}{d\mu_1}\right] = \begin{cases} 0 & \text{if } 0<x<1, \\ \dfrac{1}{\sum_{n=1}^{\infty} \frac{x^3}{n^3}\chi_{[n,n+1)}(x)} & \text{if } 1 \le x < \infty \end{cases}$$

$$= \left(\frac{[\![x]\!]}{x}\right)^3 \quad \text{if} \quad 0<x<\infty.$$

(Here $[\![x]\!]$ denotes the greatest integer not exceeding x.)

$\boxed{\#4.}$ Lemma: Let $E \subseteq \mathbb{R}$ have positive Lebesgue measure. Then
10 $E+E$ contains a nonempty open interval I.

Proof: This can be established using the same type of argument as the Steinhaus difference set lemma, as given in the lecture notes. We present an argument of a different nature which anticipates the Tonelli Theorem (Theorem 12.20).

Without loss of generality, we may assume that $0 < m(E) < \infty$, so $\chi_E \in L^2(\mathbb{R})$. Then the convolution $\chi_E * \chi_E$ is a continuous function on $\mathbb{R}$ (see the solution to problem C on HW Set #3) satisfying

$$\begin{aligned}
\int_{\mathbb{R}} (\chi_E * \chi_E)(x)\,dx &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_E(t)\chi_E(x-t)\,dt \right) dx \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_E(t)\chi_E(x-t)\,dx \right) dt \qquad \text{(via Tonelli)} \\
&= \int_{\mathbb{R}} \chi_E(t) \left(\int_{\mathbb{R}} \chi_E(x-t)\,dx \right) dt \\
&= \int_{\mathbb{R}} \chi_E(t)\, m(E+t)\,dt \\
&= \int_{\mathbb{R}} \chi_E(t)\, m(E)\,dt \qquad \text{(translation invariance of Lebesgue measure)} \\
&= (m(E))^2 \\
&> 0 .
\end{aligned}$$

Since $\chi_E * \chi_E \geq 0$, there exists $x_0 \in \mathbb{R}$ such that $(\chi_E * \chi_E)(x_0) > 0$. Continuity of $\chi_E * \chi_E$ implies the existence of $\delta > 0$ such that $(\chi_E * \chi_E)(x) > 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$. But

$$0 < (\chi_E * \chi_E)(x) = \int_{\mathbb{R}} \chi_E(t)\chi_E(x-t)\,dt = \int_{\mathbb{R}} \chi_E(t)\chi_{x-E}(t)\,dt = \int_{\mathbb{R}} \chi_{E\cap(x-E)}(t)\,dt$$
$$= m\big(E \cap (x-E)\big)$$

implies $E \cap (x-E) \neq \phi$, so to each $x \in (x_0-\delta, x_0+\delta)$ there correspond $e_x \in E$ and $f_x \in E$ such that $e_x = x - f_x$. That is, $x = e_x + f_x \in E+E$ so $(x_0 - \delta, x_0 + \delta) \subseteq E+E$. Q.E.D.

Solution of #4. Let $E \subseteq \mathbb{T}$ have positive Lebesgue measure. Then by the Lemma

10 there exists an interval $I = (a,b) \subseteq E+E$ with $b > a$. Thus $(2a, 2b) = (a,b) + (a,b) \subseteq (E+E) + (E+E)$ and, in general,

10 $(Na, Nb) \subseteq (E+E) + (E+E) + \dots + (E+E)$ (N times) for all positive integers N. If we choose N sufficiently large that $N(b-a) > 1$ then $E+E+\dots+E$ ($2N$ times) contains an interval of length

10 greater than one in $\mathbb{R}$. Reducing modulo one we have

$$\underbrace{E+E+\dots+E}_{2N \text{ times}} \pmod 1 \supseteq [0,1).$$

10 That is $\underbrace{E+E+\dots+E}_{2N \text{ times}} = \mathbb{T}$ and E is a basis for $\mathbb{T}$.

$\boxed{\#5.}$ Let χ be a measurable character of $\mathbb{T}$. Since χ maps $\mathbb{T}$ into the nonzero complex numbers, $\mathbb{C}^*$, and $\mathbb{C}^* = \bigcup_{m=1}^{\infty} A_m$ where $A_m = \{ z \in \mathbb{C} : \frac{1}{m} \le |z| \le m \}$, it follows that

$$\mathbb{T} = \chi^{-1}(\mathbb{C}^*) = \bigcup_{m=1}^{\infty} \chi^{-1}(A_m).$$

Since $\mathbb{T}$ has Lebesgue measure 1, it cannot be a countable union of sets of measure zero. Therefore, there exists an integer $m_0 \ge 1$ such that $\chi^{-1}(A_{m_0})$ has positive Lebesgue measure. By problem #4 there exists an integer $N \ge 1$ such that

$$\mathbb{T} = \chi^{-1}(A_{m_0}) + \chi^{-1}(A_{m_0}) + \dots + \chi^{-1}(A_{m_0}) \qquad (N \text{ times})$$

That is, for each $t \in \mathbb{T}$ there correspond N points $t_1, t_2, \dots, t_N$ in $\mathbb{T}$ such that $t = t_1 + t_2 + \dots + t_N$ and $\frac{1}{m_0} \le |\chi(t_i)| \le m_0$ for all $1 \le i \le N$. Consequently,

$$|\chi(t)| = |\chi(t_1 + t_2 + \dots + t_N)| = |\chi(t_1)\chi(t_2)\cdots\chi(t_N)| \in \left[\frac{1}{m_0^N}, m_0^N \right].$$

In other words, $\chi[\mathbb{T}] \subseteq \left[\frac{1}{m_0^N}, m_0^N \right]$.

Suppose that there exists $t \in \mathbb{T}$ such that $|\chi(t)| \equiv a > 1$. Then $|\chi(t+t)| = |\chi(t)\chi(t)| = a^2$ and, in general, $|\chi(\sum_{i=1}^{n} t)| = a^n$ for any positive integer n. Since $a > 1$ implies $\lim_{n\to\infty} a^n = \infty$, this contradicts the fact that x is bounded above on $\mathbb{T}$. A similar argument shows that there exists no $t \in \mathbb{T}$ such that $|x(t)| \equiv b < 1$. Consequently, $|x(t)| = 1$ for all $t \in \mathbb{T}$; i.e. x maps $\mathbb{T}$ into $\mathbb{T}^* = \{ z \in \mathbb{C} : |z| = 1 \}$.

$\boxed{\#6.}$ Let x be a measurable character of $\mathbb{T}$. Since x is a bounded measurable function on $\mathbb{T}$ (see problem #5), $x \in L^1(\mathbb{T})$ and the Fourier transform of x is defined:

$$\hat{x}(n) = \int_0^1 x(t)e^{-2\pi i n t}\, dm \qquad (n \in \mathbb{Z}).$$

Let $n \in \mathbb{Z}$ and $\tau \in \mathbb{T}$. Then translation invariance of Lebesgue measure on $\mathbb{T}$ implies

$$\hat{x}(n) = \int_0^1 x(t-\tau)e^{-2\pi i n(t-\tau)}\, dm$$

$$= \int_0^1 x(t)x(-\tau)e^{-2\pi i n t} \cdot e^{2\pi i n \tau}\, dm$$

$$= x(-\tau)e^{2\pi i n \tau}\hat{x}(n).$$

Suppose that $\hat{x}(n) \neq 0$ for some integer n. Then

$$\hat{x}(n)\left[1 - x(-\tau)e^{2\pi i n \tau}\right] = 0 \qquad \text{for all } \tau \in \mathbb{T}$$

implies $1 = x(-\tau)e^{2\pi i n \tau}$ for all $\tau \in \mathbb{T}$, and hence

$$x(\tau) = e^{2\pi i n \tau} \qquad (\tau \in \mathbb{T}).$$

It follows that, for all integers $k \neq n$,

$$\hat{x}(k) = \int_0^1 x(t)e^{-2\pi i k t} = \int_0^1 e^{2\pi i n t} \cdot e^{-2\pi i k t}\, dt = 0.$$