Mathematics 415 Final Exam, Fall 2003 Name:________________

This is an open-book, open-notes examination. You will have two hours to complete your solutions. Solve any four of the following seven problems. Each problem has the same point value ... 75 points. **CIRCLE** the numbers of the four problems which you wish me to grade.

1. Let A and B be subsets of $\mathbf{R}$ with positive Lebesgue measure. Show that
$$A+B=\{a+b:\ a\in A, b\in B\}$$
contains a nonempty open interval.

2. Let E be a Lebesgue measurable subset of $\mathbf{R}$. The metric density of E at a real number x is defined to be
$$\lim_{\varepsilon\to 0^+}\frac{m(E\cap(x-\varepsilon,x+\varepsilon))}{2\varepsilon},$$
provided the limit exists. Show that the metric density of E is 1 for almost every point of E and is 0 for almost every point of the complement of E.

3. Let $1\le p<\infty$ and, for $f\in L^p(0,\infty)$, define
$$F(x)=\frac{1}{x}\int_0^x f(t)dt \qquad (0<x<\infty).$$

(a) If $f\ge 0$ and $f\in L^p(0,\infty)$ for some $1<p<\infty$, show that
$$\int_0^\infty F^p(x)dx=-p\int_0^\infty F^{p-1}(x)xF'(x)dx=-p\int_0^\infty F^{p-1}(x)(f(x)-F(x))dx.$$

(b) If $f\in L^p(0,\infty)$ for some $1<p<\infty$, show that $\|F\|_p\le\frac{p}{p-1}\|f\|_p$.

(c) Show that the inequality in part (b) is sharp by considering the function
$$f(x)=\begin{cases}x^{-1/p} & \text{if } 1\le x\le A,\\ 0 & \text{otherwise,}\end{cases}$$
for large A.

(d) If $f\ge 0$ and $f\in L^1(0,\infty)$, show that it is possible that $F\notin L^1(0,\infty)$.

4. Let $f\in L^p(0,1)$ for some $p>0$. For $t\ge 0$, let $E_t=\{x\in(0,1):\ |f(x)|>t\}$ and let $m_f(t)$ denote the Lebesgue measure of the set E_t.

(a) Show that $h(x,t)=t^{p-1}\chi_{E_t}(x)$ is a measurable function on $(0,1)\times(0,\infty)$.

(b) Show that $\int_0^1|f(x)|^p\,dx=p\int_0^\infty t^{p-1}m_f(t)dt$.

(c) Use part (b) to show that $f\in L^p(0,1)$ if and only if $\sum_{n=1}^\infty m(\{x\in(0,1):|f(x)|^p\ge n\})<\infty$.

5. In this problem, define $\ln(0)$ as $-\infty$ and $\exp(-\infty)$ as 0, and let $f\in L^p(0,1)$ for all $p>0$.

(a) Show that $\ln(t)\le t-1$ if $0\le t<\infty$.

(b) Replace t by $|f(x)|/\|f\|_1$ in the inequality of part (a) and integrate over (0,1). What inequality results?

(c) Show that for each $t \in [0,\infty)$, $\dfrac{t^r - 1}{r}$ decreases to $\ln(t)$ as $r \to 0^+$.

(d) Is is true that $\lim_{r\to 0^+} \dfrac{\int_0^1 |f(x)|^r \, dx - 1}{r} = \int_0^1 \ln|f(x)|dx$? Explain.

(e) Prove or disprove: $\lim_{r\to 0^+} \|f\|_r = \exp\left(\int_0^1 \ln|f(x)|dx\right)$.

6. Let m denote Lebesgue measure on $(0,\infty)$. For any Lebesgue measurable subset E of $(0,\infty)$, define

$$\mu_1(E) = \sum_{n=1}^{\infty} \frac{1}{n^3} \int_{E\cap[n,n+1)} x dm,$$

$$\mu_2(E) = \int_{E\cap[1,\infty)} \frac{1}{x^2} dm.$$

Is m absolutely continuous with respect to μ_2? Is μ_2 absolutely continuous with respect to μ_1? Explain why or why not, and find the corresponding Radon-Nikodym derivatives, if they exist.

7. Let (X,Σ,μ) be a finite measure space, and S denote the set of (equivalence classes of) measurable real functions on X. (As usual, we will say that two real measurable functions on X are equivalent if they agree almost everywhere with respect to μ.) For f and g in S, define

$$d(f,g) = \int_X \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} d\mu(x).$$

Show that d is a metric on S and that $f_n \to f$ in this metric if and only if $f_n \to f$ in measure.