1. Let $A$ and $B$ be subsets of $\mathbb{R}$ with positive Lebesgue measure. Show that $A+B = \{a+b : a \in A, b \in B\}$ contains a nonempty open interval.

2. Let $E$ be a Lebesgue measurable subset of $\mathbb{R}$. The metric density of $E$ at a real number $x$ is defined to be

$$\lim_{\varepsilon \to 0} \frac{m(E \cap (x-\varepsilon,x+\varepsilon))}{2\varepsilon},$$

provided the limit exists. Show that the metric density of $E$ is 1 for almost every point of $E$ and is 0 for almost every point of the complement of $E$.

3. Let $1 \leq p < \infty$ and, for $f \in L^p(0,\infty)$, define

$$F(x) = \frac{1}{x} \int_0^x f(t) dt \quad (0 < x < \infty).$$

(a) If $f \geq 0$ and $f \in L^p(0,\infty)$ for some $1 < p < \infty$, show that

$$\int_0^x F^p(x) dx = -p \int_0^x F^{p-1}(x) x F'(x) dx = -p \int_0^x F^{p-1}(x)(f(x) - F(x)) dx.$$

(b) If $f \in L^p(0,\infty)$ for some $1 < p < \infty$, show that $\|F\|_p \leq \frac{p}{p-1} \|f\|_p$.

(c) Show that the inequality in part (b) is sharp by considering the function

$$f(x) = \begin{cases} x^{-\frac{1}{p}} & \text{if } 1 \leq x \leq A, \\ 0 & \text{otherwise}, \end{cases}$$

for large $A$.

(d) If $f \geq 0$ and $f \in L^1(0,\infty)$, show that it is possible that $F \notin L^1(0,\infty)$.

4. Let $f \in L^p(0,1)$ for some $p > 0$. For $t \geq 0$, let $E_t = \{x \in (0,1) : |f(x)| > t\}$ and let $m_r(t)$ denote the Lebesgue measure of the set $E_t$.

(a) Show that $h(x,t) = t^{-p} x^p_0$ is a measurable function on $(0,1) \times (0,\infty)$.

(b) Show that $\int_0^1 |f(x)|^p dx = p \int_0^\infty m_r(t) dt$.

(c) Use part (b) to show that $f \in L^p(0,1)$ if and only if $\sum_{r=1}^\infty m(\{x \in (0,1) : |f(x)|^p \geq r\}) < \infty$.

5. In this problem, define $\ln(0)$ as $-\infty$ and $\exp(-\infty)$ as 0, and let $f \in L^p(0,1)$ for all $p > 0$.

(a) Show that $\ln(t) \leq t-1$ if $0 \leq t < \infty$.

(b) Replace $t$ by $|f(x)|/\|f\|_p$ in the inequality of part (a) and integrate over $(0,1)$. What inequality
results?

(c) Show that for each \( t \in [0, \infty) \), \( \frac{t^r - 1}{r} \) decreases to \( \ln(t) \) as \( r \to 0^+ \).

(d) Is it true that \( \lim_{r \to 0^+} \frac{\int_0^1 |f(x)|^r \, dx - 1}{r} = \int_0^1 \ln |f(x)| \, dx \)? Explain.

(e) Prove or disprove: \( \lim_{r \to 0^+} \|f\|_r = \exp \left( \int_0^1 \ln |f(x)| \, dx \right) \).

6. Let \( m \) denote Lebesgue measure on \( (0, \infty) \). For any Lebesgue measurable subset \( E \) of \( (0, \infty) \), define

\[
\mu_1(E) = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{E \cap (n,n+1]} x \, dm,
\]

\[
\mu_2(E) = \int_{E \cap (0,\infty)} \frac{1}{x} \, dm.
\]

Is \( m \) absolutely continuous with respect to \( \mu_2 \)? Is \( \mu_2 \) absolutely continuous with respect to \( \mu_1 \)? Explain why or why not, and find the corresponding Radon-Nikodym derivatives, if they exist.

7. Let \( (X, \Sigma, \mu) \) be a finite measure space, and \( S \) denote the set of (equivalence classes of) measurable real functions on \( X \). (As usual, we will say that two real measurable functions on \( X \) are equivalent if they agree almost everywhere with respect to \( \mu \).) For \( f \) and \( g \) in \( S \), define

\[
d(f, g) = \int \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} \, d\mu(x).
\]

Show that \( d \) is a metric on \( S \) and that \( f_n \to f \) in this metric if and only if \( f_n \to f \) in measure.