Mathematics 325 Final Exam Summer 2001

This examination consists of six problems of equal value. You have the option of making this exam count either 200 or 300 points. Before turning in this exam paper, please indicate clearly your selection below.

I want this examination to count _____ points. Name: Dr. Grow

You may find the following integral useful on this exam:

$$\int \frac{dz}{a + b\cos(z)} = \frac{2}{\sqrt{a^2 - b^2}} \arctan \frac{(a - b)\tan(z/2)}{\sqrt{a^2 - b^2}} \qquad (\text{if } a^2 > b^2).$$

1. Find the solution to $xu_t - tu_x = 0$ in the xt-plane which satisfies the auxiliary condition $u(x,x) = 12x^4$ for $-\infty < x < \infty$. Sketch some characteristic curves of the partial differential equation.

2. Classify the partial differential equation

$$u_{xx} + 2u_{tt} - 3u_{xt} + u_x - 2u_t = 0$$

as hyperbolic, elliptic, parabolic, or none of these. Find the general solution in the xt-plane, if possible.

3. Consider a thin metal rod of length 1, insulated along its sides but not at its ends, which initially is at temperature 25. Suddenly both ends are plunged into a bath of temperature 0.

(a) Write the partial differential equation, boundary conditions, and initial condition that govern the temperature of the rod.

(b) Find a formula for the temperature $u(x,t)$ of the rod at position x in $[0,1]$ and at time $t \geq 0$.

4. Let ϕ be an absolutely integrable function on $-\infty < x < \infty$. Use Fourier transform methods to solve

$$u_t - u_{xx} + 2tu = 0 \qquad \text{for } -\infty < x < \infty,\ 0 < t < \infty,$$

subject to $u(x,0) = \phi(x)$ for $-\infty < x < \infty$.

5. Consider an infinite string with linear density $\rho = 1$ and tension $T = 1$, initially occupying the position of the x-axis. At time $t = 0$ and at general horizontal position x, the string is displaced vertically by an amount $\dfrac{1}{1 + x^2}$ and released with vertical velocity $\dfrac{2x}{(1 + x^2)^2}$.

(a) Write the partial differential equation and initial conditions that govern the motion of the string.

(b) Find the vertical displacement of the string as a function of position x and time t, and simplify your formula as much as possible.

(c) Show that the string moves as a standing wave to the right along the x-axis by sketching the profiles of the solution at the instants $t = 1$, 2, and 3. What is the speed of this standing wave?

(d) If instead, the string is initially displaced vertically by an amount $f(x)$ and released with vertical velocity $g(x)$ at a general horizontal position x, find the most general conditions on f and g that will produce a standing wave which moves to the right along the x-axis. (Please assume that f and g are C^2 and C^1 functions, respectively, on the real line.)

6. Let h be a piecewise smooth 2π-periodic function on the real line. Consider the Dirichlet problem for the exterior of the unit disk:

$$\nabla^2 u = 0 \qquad \text{for } r > 1,$$
$$u(1;\theta) = h(\theta) \qquad \text{for } -\pi \le \theta < \pi,$$
$$u \text{ bounded as } r \to \infty.$$

(a) Use separation of variables to show that for $r > 1$, the solution can be expressed as

$$u(r;\theta) = \sum_{n = \infty}^{\infty} \hat{h}(n) r^{-|n|} e^{in\theta}$$

where

$$\hat{h}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\phi) e^{-in\phi} d\phi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

(b) Assuming the validity of part (a), show that the solution can be expressed in the form

$$u(r;\theta) = \frac{r^2 - 1}{2\pi} \int_{-\pi}^{\pi} \frac{h(\phi)}{1 - 2r\cos(\theta - \phi) + r^2} d\phi$$

for $r > 1$.

(c) Assuming the validity of parts (a) and (b), find the solution to the problem if h is defined on a fundamental period by $h(\theta) = 0$ if $-\pi \le \theta < 0$ and $h(\theta) = 1$ if $0 \le \theta < \pi$. (For full credit, you must evaluate any infinite sum(s) or integral(s) in your answer.)

A Brief Table of Fourier Transforms

$f(x)$	$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$
A. $\begin{cases} 1 & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b\xi)}{\xi}$
B. $\begin{cases} 1 & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{-ic\xi} - e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C. $\frac{1}{x^2 + a^2}$ $(a > 0)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\|\xi\|}}{a}$
D. $\begin{cases} x & \text{if } 0 < x \le b, \\ 2b - x & \text{if } b < x < 2b, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{-1 + 2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E. $\begin{cases} e^{-ax} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$ $(a > 0)$	$\frac{1}{(a + i\xi)\sqrt{2\pi}}$
F. $\begin{cases} e^{ax} & \text{if } b < x < c, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{(a-i\xi)c} - e^{(a-i\xi)b}}{(a - i\xi)\sqrt{2\pi}}$
G. $\begin{cases} e^{iax} & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b(\xi-a))}{\xi - a}$
H. $\begin{cases} e^{iax} & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{i}{\sqrt{2\pi}} \frac{e^{ic(a-\xi)} - e^{id(a-\xi)}}{a - \xi}$
I. e^{-ax^2} $(a > 0)$	$\frac{1}{\sqrt{2a}} e^{-\xi^2/(4a)}$
J. $\frac{\sin(ax)}{x}$ $(a > 0)$	$\begin{cases} 0 & \text{if } \|\xi\| \ge a, \\ \sqrt{\pi/2} & \text{if } \|\xi\| < a. \end{cases}$

Convergence Theorems

$$X'' + \lambda X = 0 \text{ in } (a, b) \text{ with any } \textit{symmetric} \text{ BC.} \tag{1}$$

Now let $f(x)$ be any function defined on $a \le x \le b$. Consider the Fourier series for the problem (1) with any given boundary conditions that are symmetric. We now state a convergence theorem for each of the three modes of convergence. They are partly proved in the next section.

Theorem 2. Uniform Convergence The Fourier series $\Sigma A_n X_n(x)$ converges to $f(x)$ uniformly on $[a, b]$ provided that

(i) $f(x)$, $f'(x)$, and $f''(x)$ exist and are continuous for $a \le x \le b$ and

(ii) $f(x)$ satisfies the given boundary conditions.

Theorem 3. L^2 Convergence The Fourier series converges to $f(x)$ in the mean-square sense in (a, b) provided only that $f(x)$ is any function for which

$$\int_a^b |f(x)|^2 \, dx \text{ is finite.} \tag{8}$$

Theorem 4. Pointwise Convergence of Classical Fourier Series

(i) The classical Fourier series (full or sine or cosine) converges to $f(x)$ pointwise on (a, b), provided that $f(x)$ is a continuous function on $a \le x \le b$ and $f'(x)$ is piecewise continuous on $a \le x \le b$.

(ii) More generally, if $f(x)$ itself is only piecewise continuous on $a \le x \le b$ and $f'(x)$ is also piecewise continuous on $a \le x \le b$, then the classical Fourier series converges at every point x $(-\infty < x < \infty)$. The sum is

$$\sum_n A_n X_n(x) = \tfrac{1}{2}[f(x+) + f(x-)] \qquad \text{for all } a < x < b. \tag{9}$$

The sum is $\frac{1}{2}[f_{\text{ext}}(x+) + f_{\text{ext}}(x-)]$ for all $-\infty < x < \infty$, where $f_{\text{ext}}(x)$ is the extended function (periodic, odd periodic, or even periodic).

Theorem 4∞. If $f(x)$ is a function of period $2l$ on the line for which $f(x)$ and $f'(x)$ are piecewise continuous, then the classical full Fourier series converges to $\frac{1}{2}[f(x+) + f(x-)]$ for $-\infty < x < \infty$.

#1 $x u_t - t u_x = 0 \longleftrightarrow (-t,x)\cdot\nabla u = 0 \longleftrightarrow D_{(-t,x)}\, u = 0$.

Therefore u remains constant along curves whose tangent is parallel to $(-t,x)$, i.e.

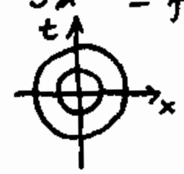
13 those satisfying $\frac{dt}{dx} = \frac{x}{-t} \Rightarrow -t\,dt = x\,dx \Rightarrow -\frac{t^2}{2} + c = \frac{x^2}{2} \Rightarrow c_1 = x^2 + t^2$.

Along such a curve $u(x,t) = u(x, \sqrt{c_1 - x^2}\,) = u(0, \sqrt{c_1 - 0}\,) = f(c_1)$. Thus

24 the general solution in the plane is $u(x,t) = f(x^2+t^2)$ where f is a C^1 function

of a single real variable. $12x^4 = u(x,x) = f(2x^2) \longleftrightarrow 3\bigstar^2 = f(\bigstar)$.

32 Consequently, $\boxed{u(x,t) = 3(x^2+t^2)^2}$ solves the I.V.P. 3



Characteristic curves (circles

5 **#2** $B^2 - 4AC = (-3)^2 - 4(1)(2) = 1 > 0$. The equation is $\boxed{\text{hyperbolic.}}$

$$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x \partial t} + 2\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial t} = 0 \longleftrightarrow \left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)u + \left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t}\right)u = 0$$

15 Let $\begin{cases} \xi = 2x + t, \\ \eta = x + t. \end{cases}$ Then the chain rule gives $\frac{\partial}{\partial x} = 2\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$,

so $\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)u + \left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t}\right)u = 0 \longleftrightarrow \left[\cancel{2\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta} - 2\left(\cancel{\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta}\right)\right]\left[2\frac{\partial}{\partial \xi} + \cancel{\frac{\partial}{\partial \eta}} -\right.$

23 $\left.\left(\frac{\partial}{\partial \xi} + \cancel{\frac{\partial}{\partial \eta}}\right)\right]u + \left[\cancel{2\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta} - 2\left(\cancel{\frac{\partial}{\partial \xi}} + \frac{\partial}{\partial \eta}\right)\right]u = 0 \longleftrightarrow -\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right) - \frac{\partial u}{\partial \eta} = 0$

$\longleftrightarrow \frac{\partial v}{\partial \xi} + v = 0$ where $v = \frac{\partial u}{\partial \eta}$. Integrating once w.r.t. ξ gives $\left(\frac{\partial u}{\partial \eta} =\right) v = c_1(\eta)e^{-\xi}$

and integrating again w.r.t. η gives $u = f(\eta)e^{-\xi} + g(\xi)$. Therefore, the

33 general solution in the xt-plane is $\boxed{u(x,t) = f(x+t)e^{-2x-t} + g(2x+t)}$

where f and g are arbitrary C^2 functions of a single real variable.

#3 (a)

9 $$\begin{cases} u_t - k u_{xx} = 0 & \text{for } 0<x<1,\ 0<t<\infty, \quad (1) \\ u(0,t) = u(1,t) = 0 & \text{for } 0<t<\infty, \quad (2)\text{-}(3) \\ u(x,0) = 25 & \text{for } 0<x<1. \quad (4) \end{cases}$$

(b) Let $u(x,t) = X(x)T(t)$ be a $\overset{\text{nontrivial}}{\text{solution}}$ of (1)-(2)-(3). Then $X(x)T'(t) - kX''(x)T(t) = 0$

$\Rightarrow \ -\dfrac{X''(x)}{X(x)} = -\dfrac{T'(t)}{kT(t)} = \text{constant} = \lambda$. Applying (2)-(3) yields the coupled system

20 $$\begin{cases} X''(x) + \lambda X(x) = 0, \quad X(0) = 0 = X(1). \\ T'(t) + k\lambda T(t) = 0 \end{cases}$$

Eigenvalues	Eigenfunctions
$\lambda_n = (n\pi)^2$	$X_n(x) = \sin(n\pi x)$
$(n = 1, 2, 3, \dots)$	

Also $T_n(t) = e^{-k(n\pi)^2 t}$ (up to a constant factor), so a formal solution of (1)-(2)-(3) is

24 $$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\, e^{-k(n\pi)^2 t}.$$

$$25 = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \text{ for } 0<x<1 \ \Rightarrow\ b_n = \frac{\langle 25, \sin(n\pi\,\cdot)\rangle}{\langle \sin(n\pi\,\cdot), \sin(n\pi\,\cdot)\rangle} = 50\int_0^1 \sin(n\pi x)\,dx$$

$$= \left.\frac{-50\cos(n\pi x)}{n\pi}\right|_0^1 = \frac{50 - 50(-1)^n}{n\pi} = \begin{cases} 0 & \text{if } n = 2m \text{ is even}, \\ \dfrac{100}{(2m-1)\pi} & \text{if } n = 2m-1 \text{ is odd}. \end{cases}$$

33 $$\therefore \quad \boxed{u(x,t) = \frac{100}{\pi}\sum_{m=1}^{\infty} \frac{\sin((2m-1)\pi x)\, e^{-k(2m-1)^2\pi^2 t}}{2m-1}}.$$

#4 $\mathcal{F}\left(u_t - u_{xx} + 2tu\right)(\xi) = \mathcal{F}(0)(\xi) = 0$

+0 $$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) - (i\xi)^2\mathcal{F}(u)(\xi) + 2t\mathcal{F}(u)(\xi) = 0$$

$$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + (\xi^2+2t)\mathcal{F}(u)(\xi) = 0 \quad \longleftarrow \text{ Integrating factor: } e^{\int(\xi^2+2t)dt} = e^{\xi^2 t + t^2}.$$

$$e^{\xi^2 t+t^2}\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + (\xi^2+2t)e^{\xi^2 t+t^2}\mathcal{F}(u)(\xi) = 0 \quad \longleftrightarrow \quad \frac{\partial}{\partial t}\left[e^{\xi^2 t+t^2}\mathcal{F}(u)(\xi)\right] = 0$$

$\Rightarrow e^{\xi^2 t+t^2}\mathcal{F}(u)(\xi) = c_1(\xi) \Rightarrow \mathcal{F}(u)(\xi) = c_1(\xi)e^{-\xi^2 t - t^2}$. Applying the initial condition yields $c_1(\xi) = c_1(\xi)e^{-\xi^2 t-t^2}\Big|_{t=0} = \mathcal{F}(u)(\xi)\Big|_{t=0} = \mathcal{F}(u(\cdot,0))(\xi) = \mathcal{F}(\varphi)(\xi)$.

+21 Thus, $\mathcal{F}(u)(\xi) = \mathcal{F}(\varphi)(\xi)e^{-\xi^2 t - t^2}$. Applying formula I in the table of Fourier transforms with $a = 1/4t$ gives $\mathcal{F}\left(\frac{1}{\sqrt{2t}}e^{-(\cdot)^2/4t}\right)(\xi) = e^{-\xi^2 t}$ so

$$\mathcal{F}(u)(\xi) = \mathcal{F}(\varphi)(\xi)e^{-\xi^2 t}e^{-t^2} = \mathcal{F}(\varphi)(\xi)\mathcal{F}\left(\frac{e^{-t^2}}{\sqrt{2t}}e^{-(\cdot)^2/4t}\right)(\xi)$$

$$= \frac{1}{\sqrt{2\pi}}\mathcal{F}\left(\varphi * \frac{e^{-t^2}}{\sqrt{2t}}e^{-(\cdot)^2/4t}\right)(\xi)$$

+28 $$= \mathcal{F}\left(\varphi * \frac{e^{-t^2}}{\sqrt{4\pi t}}e^{-(\cdot)^2/4t}\right)(\xi).$$

The inversion theorem yields

+30 $$u(x,t) = \left(\varphi * \frac{e^{-t^2}}{\sqrt{4\pi t}}e^{-(\cdot)^2/4t}\right)(x)$$

$$= \int_{-\infty}^{\infty}\frac{e^{-t^2}}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\varphi(y)\,dy$$

+35 $$= \boxed{\frac{e^{-t^2}}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}e^{-\frac{(x-y)^2}{4t}}\varphi(y)\,dy} \qquad (\text{at least for } t>0).$$

#5 (a)

+10 $$\begin{cases} u_{tt} - u_{xx} = 0 & \text{for } -\infty < x < \infty,\ 0 < t < \infty, \\ u(x,0) = \dfrac{1}{1+x^2} \text{ and } u_t(x,0) = \dfrac{2x}{(1+x^2)^2} & \text{for } -\infty < x < \infty. \end{cases}$$

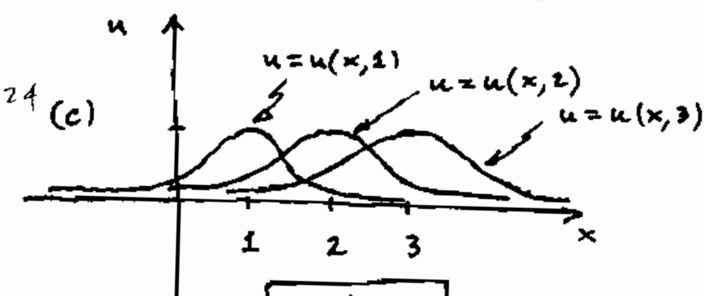
(b) $$u(x,t) = \frac{1}{2}\Big[f(x+t) + f(x-t)\Big] + \frac{1}{2}\int_{x-t}^{x+t} g(\xi)\,d\xi \qquad (\text{d'Alembert})$$

+20 $$= \frac{1}{2}\left[\frac{1}{1+(x+t)^2} + \frac{1}{1+(x-t)^2}\right] + \frac{1}{2}\int_{x-t}^{x+t} \frac{2\xi}{(1+\xi^2)^2}\,d\xi \longleftarrow \left\{ \begin{array}{l} \text{Let } w = 1+\xi^2 \\ \text{Then } dw = 2\xi\,d\xi \\ \int \frac{dw}{w^2} = -\frac{1}{w} \end{array} \right.$$

$$= \frac{1}{2}\left[\cancel{\frac{1}{1+(x+t)^2}} + \frac{1}{1+(x-t)^2}\right] + \frac{1}{2}\left[\cancel{\frac{-1}{1+(x+t)^2}} + \frac{1}{1+(x-t)^2}\right]$$

$$= \boxed{\frac{1}{1+(x-t)^2}}$$

+24 (c)



$\boxed{\text{speed} = 1}$

(d) In order for the solution to be a standing wave that moves to the right along the x-axis, we need u to be a function of $x-t$ only. Thus, comparing with the d'Alembert formula (see part (b)), we must have

$$\frac{1}{2} f(x+t) + \frac{1}{2}\int_0^{x+t} g(\xi)\,d\xi = 0 \quad \text{for all } -\infty < x < \infty \text{ and all } 0 < t < \infty.$$

33 I.e. $\boxed{f(z) + \int_0^z g(\xi)\,d\xi = 0 \quad \text{for all real } z.}$ Differentiating with respect to z, we get the equivalent condition $\boxed{f'(z) = -g(z) \quad \text{for all real } z.}$

#6

(1: (a)
$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & \text{for } r > 1, \quad ① \\ u \text{ bounded as } r \to \infty, & ② \\ u(1;\theta) = h(\theta) & \text{for } -\pi \le \theta < \pi. \quad ③ \end{cases}$$

Let $u(r;\theta) = R(r)\Theta(\theta)$ be a nontrivial solution of ①-②. Then

$$r^2\left[R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta)\right] = 0 \cdot r^2$$

$$\Rightarrow \quad \frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \text{constant} = \lambda .$$

Since $(r;\pm\pi)$ correspond to the same point in the plane for each $r > 1$, the function Θ must satisfy periodic boundary conditions. Hence

+4
$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0, \quad \Theta(-\pi) = \Theta(\pi), \quad \Theta'(-\pi) = \Theta'(\pi) \\ r^2R''(r) + rR'(r) - \lambda R(r) = 0 \end{cases}$$

Eigenvalues: $\lambda_n = n^2 \quad (n = 0, 1, 2, \ldots)$

+6
Eigenfunctions: $\Theta_n(\theta) = \alpha_n e^{in\theta} + \beta_n e^{-in\theta} \quad (n = 0, 1, 2, \ldots)$

The n^{th} radial solution satisfies

$$r^2R_n''(r) + rR_n'(r) - n^2R_n(r) = 0 \qquad \text{(Euler-Cauchy)}.$$

$R_n(r) = r^\gamma$ where γ is a constant to be determined. Then $R_n'(r) = \gamma r^{\gamma-1}$ and $R_n''(r) = \gamma(\gamma-1)r^{\gamma-2}$ so $r^2\left(\gamma(\gamma-1)r^{\gamma-2}\right) + r\left(\gamma r^{\gamma-1}\right) - n^2r^\gamma = 0$

$$\Rightarrow \quad \gamma(\gamma-1) + \gamma - n^2 = 0$$

$$\Rightarrow \quad \gamma^2 - n^2 = 0 \qquad \Rightarrow \quad \gamma = \pm n$$

$\therefore$ if $n \ge 1$, the general solution to the radial equation is $R_n(r) = c_1r^n + c_2r^{-n}$.

+8
If $n = 0$, the general solution to the radial equation is $R_0(r) = c_1\ln(r) + c_2$.

In order that $u_n(r;\theta) = R_n(r)\Theta_n(\theta)$ satisfy ②, we must have $c_1 = 0$ in each case $n=0,1,2,\dots$ for the radial portion of the solution. Hence the formal solution of ①-② is

+9 $$u(r;\theta) = \sum_{n=-\infty}^{\infty} r^{-|n|} c_n e^{in\theta} .$$

In order to satisfy ③, we must have

$$h(\theta) = u(1;\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \qquad \text{for all } -\pi \le \theta < \pi .$$

Consequently,

+11 $$(*) \qquad c_n = \frac{\langle h, e^{in(\cdot)} \rangle}{\langle e^{in(\cdot)}, e^{in(\cdot)} \rangle} = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\varphi) e^{-in\varphi}\, d\varphi = \hat{h}(n)$$

for all $n = 0, \pm 1, \pm 2, \dots$ That is,

+12 $$(**) \qquad \boxed{u(r;\theta) = \sum_{n=-\infty}^{\infty} \hat{h}(n) r^{-|n|} e^{in\theta}} \qquad (r > 1).$$

(10pts) (b) From $(*)$ and $(**)$ in (a) we obtain

$$u(r;\theta) = \sum_{n=-\infty}^{\infty} r^{-|n|} e^{in\theta} \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\varphi) e^{-in\varphi}\, d\varphi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\varphi) \left\{ \sum_{n=-\infty}^{\infty} r^{-|n|} e^{in(\theta-\varphi)} \right\} d\varphi \qquad (\text{since } r > 1)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\varphi)\, P(r;\theta-\varphi)\, d\varphi$$

+4 where $P(\rho;\psi) = \sum_{n=-\infty}^{\infty} \rho^{-|n|} e^{in\psi}$ for $\rho > 1$ and $-\pi \le \psi < \pi$.

Observe that

$$P(\rho;\psi) = \sum_{n=1}^{\infty} \rho^{-n} e^{-in\psi} + 1 + \sum_{n=1}^{\infty} \rho^{-n} e^{in\psi}$$

$$= \sum_{n=1}^{\infty} (\rho^{-1} e^{-i\psi})^n + 1 + \sum_{n=1}^{\infty} (\rho^{-1} e^{i\psi})^n$$

$$= \frac{\rho^{-1} e^{-i\psi}}{1 - \rho^{-1} e^{-i\psi}} + 1 + \frac{\rho^{-1} e^{i\psi}}{1 - \rho^{-1} e^{i\psi}} \qquad (\text{since } \rho > 1)$$

$$= \frac{\rho^{-1} e^{-i\psi}(1 - \rho^{-1} e^{i\psi}) + (1 - \rho^{-1} e^{-i\psi})(1 - \rho^{-1} e^{i\psi}) + \rho^{-1} e^{i\psi}(1 - \rho^{-1} e^{-i\psi})}{(1 - \rho^{-1} e^{-i\psi})(1 - \rho^{-1} e^{i\psi})}$$

$$= \frac{\cancel{\rho^{-1} e^{-i\psi}} - \rho^{-2} + 1 + \cancel{\rho^{-2}} - \cancel{2\rho^{-1}\cos(\psi)} + \cancel{\rho^{-1} e^{i\psi}} - \cancel{\rho^{-2}}}{1 - 2\rho^{-1}\cos(\psi) + \rho^{-2}}$$

+8

$$= \frac{\rho^2 - 1}{\rho^2 - 2\rho\cos(\psi) + 1} .$$

Substituting this expression in the previous formula for the solution gives

$$u(r;\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h(\varphi)\,[r^2 - 1]}{1 - 2r\cos(\theta - \varphi) + r^2}\, d\varphi$$

+10

$$= \boxed{\frac{r^2 - 1}{2\pi} \int_{-\pi}^{\pi} \frac{h(\varphi)\, d\varphi}{1 - 2r\cos(\theta - \varphi) + r^2}} . \qquad (\text{for } r > 1)$$

(11 pts.) (c) Using the formula in part (b), we have

+2 $$u(r;\theta) = \frac{r^2-1}{2\pi}\int_0^{\pi} \frac{1\,d\varphi}{1-2r\cos(\theta-\varphi)+r^2}$$

$$\left\{\begin{array}{l} \text{Let } z = \theta-\varphi. \\ \text{Then } dz = -d\varphi. \\ \varphi=\pi \Rightarrow z = \theta-\pi. \\ \varphi = 0 \Rightarrow z = \theta. \end{array}\right.$$

+4 $$= \frac{r^2-1}{2\pi}\int_{\theta-\pi}^{\theta} \frac{1\,dz}{1+r^2-2r\cos(z)}.$$

Apply the integral formula on the first page of this exam with $a = 1+r^2$ and $b = -2r$ to get

+6 $$u(r;\theta) = \frac{r^2-1}{2\pi}\cdot\frac{2}{\sqrt{(1+r^2)^2-(-2r)^2}}\arctan \frac{(1+r^2+2r)\tan(z/2)}{\sqrt{(1+r^2)^2-(-2r)^2}}\Bigg|_{z=\theta-\pi}^{z=\theta}.$$

+8 $$= \frac{r^2-1}{\pi}\cdot\frac{1}{\sqrt{(r^2-1)^2}}\arctan\frac{(1+r)^2\tan(z/2)}{\sqrt{(r^2-1)^2}}\Bigg|_{z=\theta-\pi}^{z=\theta}$$

+10 $$= \frac{1}{\pi}\arctan\left(\frac{r+1}{r-1}\tan\left(\frac{\theta}{2}\right)\right) - \frac{1}{\pi}\arctan\left(\frac{r+1}{r-1}\tan\left(\frac{\theta}{2}-\frac{\pi}{2}\right)\right)$$

+11 $$= \boxed{\frac{1}{\pi}\arctan\left(\frac{r+1}{r-1}\tan\left(\frac{\theta}{2}\right)\right) + \frac{1}{\pi}\arctan\left(\frac{r+1}{r-1}\cot\left(\frac{\theta}{2}\right)\right)}$$