Mathematics 325 | Final Exam Summer 2002 | Name: Dr. Grow (4 pts.)

1.(28 pts.) Consider

(*) $$tu_x - xu_t = 0.$$

(a) Find the characteristic curves of (*) and sketch two.

(b) Write the general solution of (*).

(c) Determine the particular solution of (*) that satisfies the auxiliary condition $u(x,0) = \exp(-x^2)$ for $-\infty < x < \infty$.

(d) In what region of the xt-plane is the solution in part (c) uniquely determined?

2.(28 pts.) Consider

(+) $$u_{xx} + u_{yy} - 2u_{xy} + u = (x - y)^2.$$

(a) Classify the order (first, second, etc.) and type (nonlinear, linear, homogeneous, inhomogeneous, elliptic, etc.) of (+).

(b) Find, if possible, the general solution of (+) in the xy-plane.

3.(28 pts.) (a) Write and simplify an expression for the solution to

(%) $$u_{yy} - u_{xx} = 0 \quad \text{for } -\infty < x < \infty,\ 0 < y < \infty,$$

subject to the auxiliary conditions

$$u(x,0) = 1/(x^2 + 1) \quad \text{and} \quad u_y(x,0) = -2x/(x^2 + 1)^2 \quad \text{for } -\infty < x < \infty.$$

(b) Sketch profiles of the solution in part (a) at y = 1, 2, and 3 in order to illustrate that the solution is a wave traveling to the left along the x-axis. What is its speed?

(c) Derive a general nontrivial relation between ϕ and ψ which will produce a solution to (%) in the xy-plane satisfying

$$u(x,0) = \psi(x) \quad \text{and} \quad u_y(x,0) = \phi(x) \quad \text{for } -\infty < x < \infty$$

and such that u consists solely of a wave traveling to the left along the x-axis.

4.(28 pts.) Find and simplify as much as possible an expression for the solution to

$$u_y - u_{xx} = 0 \quad \text{in } -\infty < x < \infty,\ 0 < y < \infty,$$

which satisfies

$$u(x,0) = \exp(-x^2) \quad \text{for } -\infty < x < \infty.$$

5.(28 pts.) Let g be an absolutely integrable function on $(-\infty,\infty)$. Use Fourier transform methods to solve

$$u_{xx} + u_{yy} = 0 \quad \text{for } -\infty < x < \infty,\ 0 < y < \infty,$$

subject to the boundary condition

$$u(x,0) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and the decay conditions

$$\lim_{y\to\infty} u(x,y) = 0 \quad \text{for each } x \text{ in } (-\infty,\infty)$$

and

$$|u(x,y)| \le |g(x)| \quad \text{for all } x \text{ in } (-\infty,\infty) \text{ and all } y > 0.$$

6.(28 pts.) (a) Show that the operator T defined by $Tf = -f''$ is hermitian on $V = \{ f \in C^2[0,1] : f'(0) = 0,\ f(1) = 0 \}$, equipped with the standard inner product $\langle f,g \rangle = \int_0^1 f(x)\overline{g(x)}dx$.

(b) Find the eigenvalues and corresponding eigenfunctions of T on V.

(c) Is the set of functions $\{ \cos((n + 1/2)\pi x) \}_{n=0}^{\infty}$ orthogonal on $(0,1)$ with the standard inner product? Why?

(d) Show that the Fourier series representation of the function $\phi(x) = 1 - x^2$ with respect to $\{ \cos((n + 1/2)\pi x) \}_{n=0}^{\infty}$ on $[0,1]$ is

$$\phi(x) \sim \sum_{n=0}^{\infty} \frac{32(-1)^n \cos((n + 1/2)\pi x)}{(2n + 1)^3 \pi^3} .$$

(e) Discuss uniform, mean square, and pointwise convergence, or lack thereof, on $[0,1]$ for the generalized Fourier series in part (d), quoting theorems whenever appropriate to support your assertions.

(f) Find a solution to

$$u_{tt} - u_{xx} \overset{(1)}{=} 0 \quad \text{for } 0 < x < 1,\ 0 < t < \infty,$$

which satisfies

$$u_x(0,t) \overset{(2)}{=} 0,\ u(1,t) \overset{(3)}{=} 0 \quad \text{for } t \ge 0,$$

and

$$u(x,0) \overset{(5)}{=} 1 - x^2,\ u_t(x,0) \overset{(4)}{=} 0 \quad \text{for } 0 \le x \le 1.$$

Bonus (10 pts.) Is your solution to part (f) unique? Justify your answer.

7.(28 pts.) The material in a thin circular disk of radius 1 has a steady-state temperature distribution. The material is held at 50 degrees Centigrade on the top half of disk's edge and at -50 degrees Centigrade on the bottom half of its edge.

(a) Find the temperature distribution function for the material.

(b) What is the temperature of the material at the center of the disk? Justify your answer.

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$
A.	$\begin{cases} 1 & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin(b\xi)}{\xi}$
B.	$\begin{cases} 1 & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{e^{-ic\xi} - e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\dfrac{1}{x^2 + a^2}$ $\quad (a > 0)$	$\sqrt{\dfrac{\pi}{2}}\,\dfrac{e^{-a\|\xi\|}}{a}$
D.	$\begin{cases} x & \text{if } 0 < x \le b, \\ 2b - x & \text{if } b < x < 2b, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{-1 + 2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases} e^{-ax} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$ $(a > 0)$	$\dfrac{1}{(a + i\xi)\sqrt{2\pi}}$
F.	$\begin{cases} e^{ax} & \text{if } b < x < c, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{e^{(a-i\xi)c} - e^{(a-i\xi)b}}{(a - i\xi)\sqrt{2\pi}}$
G.	$\begin{cases} e^{iax} & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin(b(\xi-a))}{\xi - a}$
H.	$\begin{cases} e^{iax} & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{i}{\sqrt{2\pi}}\,\dfrac{e^{ic(a-\xi)} - e^{id(a-\xi)}}{a - \xi}$
I.	e^{-ax^2} $\quad (a > 0)$	$\dfrac{1}{\sqrt{2a}}\, e^{-\xi^2/(4a)}$
J.	$\dfrac{\sin(ax)}{x}$ $\quad (a > 0)$	$\begin{cases} 0 & \text{if } \|\xi\| \ge a, \\ \sqrt{\pi/2} & \text{if } \|\xi\| < a. \end{cases}$

Convergence Theorems

$$X'' + \lambda X = 0 \text{ in } (a, b) \text{ with any } \textit{symmetric} \text{ BC.} \qquad (1)$$

Now let $f(x)$ be any function defined on $a \leq x \leq b$. Consider the Fourier series for the problem (1) with any given boundary conditions that are symmetric. We now state a convergence theorem for each of the three modes of convergence. They are partly proved in the next section.

Theorem 2. Uniform Convergence The Fourier series $\Sigma A_n X_n(x)$ converges to $f(x)$ uniformly on $[a, b]$ provided that

(i) $f(x)$, $f'(x)$, and $f''(x)$ exist and are continuous for $a \leq x \leq b$ and

(ii) $f(x)$ satisfies the given boundary conditions.

Theorem 3. L^2 Convergence The Fourier series converges to $f(x)$ in the mean-square sense in (a, b) provided only that $f(x)$ is any function for which

$$\int_a^b |f(x)|^2 \, dx \text{ is finite.} \qquad (8)$$

Theorem 4. Pointwise Convergence of Classical Fourier Series

(i) The classical Fourier series (full or sine or cosine) converges to $f(x)$ pointwise on (a, b), provided that $f(x)$ is a continuous function on $a \leq x \leq b$ and $f'(x)$ is piecewise continuous on $a \leq x \leq b$.

(ii) More generally, if $f(x)$ itself is only piecewise continuous on $a \leq x \leq b$ and $f'(x)$ is also piecewise continuous on $a \leq x \leq b$, then the classical Fourier series converges at every point x $(-\infty < x < \infty)$. The sum is

$$\sum_n A_n X_n(x) = \tfrac{1}{2}[f(x+) + f(x-)] \qquad \text{for all } a < x < b. \qquad (9)$$

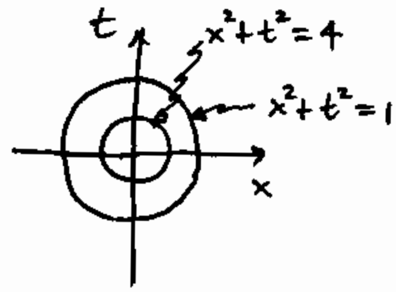
The sum is $\frac{1}{2}[f_{\text{ext}}(x+) + f_{\text{ext}}(x-)]$ for all $-\infty < x < \infty$, where $f_{\text{ext}}(x)$ is the extended function (periodic, odd periodic, or even periodic).

Theorem 4∞. If $f(x)$ is a function of period $2l$ on the line for which $f(x)$ and $f'(x)$ are piecewise continuous, then the classical full Fourier series converges to $\frac{1}{2}[f(x+) + f(x-)]$ for $-\infty < x < \infty$.

#1. $t u_x - x u_t = 0$

8 (a) Characteristic curves: $\frac{dt}{dx} = \frac{-x}{t} \Rightarrow t\,dt = -x\,dx \Rightarrow \frac{1}{2}t^2 = -\frac{x^2}{2} + c_1$

$\boxed{x^2 + t^2 = c}$



8 (b) $\therefore$ $\boxed{u(x,t) = f(x^2+t^2)}$ where f is a C^1-function of a single real variable.

8 (c) $e^{-x^2} = u(x,0) = f(x^2) \Rightarrow f(w) = e^{-w}$ for all $w \geq 0$.

$\therefore$ $\boxed{u(x,t) = e^{-(x^2+t^2)}}$

4 (d) $\boxed{\text{The solution is uniquely determined in the entire } x\text{-}t \text{ plane.}}$

t #2 (a) $\boxed{\text{linear, inhomogeneous, second order, parabolic}}$ $\left(B^2-4AC = (-2)^2 - 4(1)(1) = 0\right.$

(b) $$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + u = (x-y)^2$$

3 $$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u + u = (x-y)^2$$

6 Let $\begin{cases} \xi = x+y, \\ \eta = x-y. \end{cases}$ Then $\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$

So $\frac{\partial}{\partial x} - \frac{\partial}{\partial y} = 2\frac{\partial}{\partial \eta}$.

9 $$\left(2\frac{\partial}{\partial \eta}\right)\left(2\frac{\partial}{\partial \eta}\right)u + u = \eta^2$$

$$\frac{\partial^2 u}{\partial \eta^2} + \frac{1}{4}u = \frac{1}{4}\eta^2$$

14 $$u(\xi,\eta) = c_1(\xi)\cos\left(\frac{\eta}{2}\right) + c_2(\xi)\sin\left(\frac{\eta}{2}\right) + u_p(\xi,\eta)$$

Trial Form for particular Solution: $u_p = A\eta^2 + B\eta + C$ (Undetermined Coefficients

$$\Rightarrow \frac{\partial u_p}{\partial \eta} = 2A\eta + B \text{ and } \frac{\partial^2 u_p}{\partial \eta^2} = 2A.$$

$$\therefore \quad 2A + \frac{1}{4}(A\eta^2 + B\eta + C) = \frac{\partial^2 u_p}{\partial \eta^2} + \frac{1}{4}u_p \overset{\text{Want}}{=} \frac{1}{4}\eta^2$$

$$\Rightarrow \quad \frac{A}{4} = \frac{1}{4}, \quad \frac{B}{4} = 0, \quad 2A + \frac{C}{4} = 0 \quad \Rightarrow A = 1, B = 0, C = -8$$

19 $$\therefore \quad u_p = \eta^2 - 8$$

$$u(\xi,\eta) = c_1(\xi)\cos\left(\tfrac{1}{2}\eta\right) + c_2(\xi)\sin\left(\tfrac{1}{2}\eta\right) + \eta^2 - 8$$

22 $$\boxed{u(x,y) = f(x+y)\cos\left(\frac{x-y}{2}\right) + g(x+y)\sin\left(\frac{x-y}{2}\right) + (x-y)^2 - 8}$$

(f and g are C^2-functions of a single real variable)

16 #3 (a) $$u(x,y) = \frac{1}{2}\Big[f(x+y) + f(x-y) \Big] + \frac{1}{2}\int_{x-y}^{x+y} g(z)\,dz$$

$$= \frac{1}{2}\left[\frac{1}{(x+y)^2+1} + \frac{1}{(x-y)^2+1} \right] + \frac{1}{2}\int_{x-y}^{x+y} \frac{-2z}{(z^2+1)^2}\,dz$$

$$= \frac{1}{2}\left[\frac{1}{(x+y)^2+1} + \cancel{\frac{1}{(x-y)^2+1}} \right] + \frac{1}{2}\left[\frac{1}{(x+y)^2+1} - \cancel{\frac{1}{(x-y)^2+1}} \right]$$

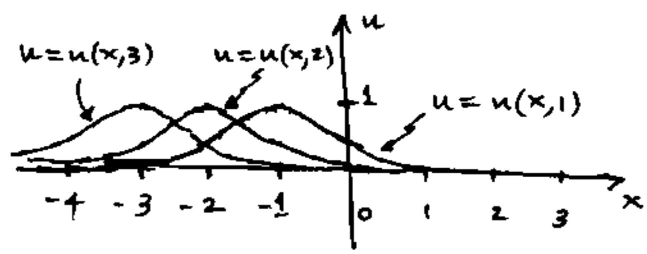
$$= \boxed{\frac{1}{(x+y)^2+1}}$$

Let $w = z^2+1$

Then $dw = 2z\,dz$

$$\therefore \int \frac{-2z\,dz}{(z^2+1)^2} = \int \frac{-dw}{w^2} = \frac{1}{w} = \frac{1}{z^2+1}$$

4 (b)



$\boxed{\text{Speed} = 1}$

8 (c) $u(x,y) = f(x+y) + g(x-y)$ (General solution of $u_{yy} - u_{xx} = 0$.)

Want $g \equiv 0$, so $u(x,y) = f(x+y)$.

$$\left.\begin{aligned} \varphi(x) &= u_y(x,0) = f'(x) \\ \psi(x) &= u(x,0) = f(x) \end{aligned}\right\} \Rightarrow \boxed{\varphi(x) = \psi'(x) \text{ for all real } x}$$

12 #4 $$u(x,y) = \frac{1}{\sqrt{4\pi y}} \int_{-\infty}^{\infty} e^{-\frac{(x-z)^2}{4y}} \cdot e^{-z^2}\, dz \qquad (y>0).$$

Complete the square (in z) in the exponent:

$$\frac{(x-z)^2}{4y} + z^2 = \frac{x^2 - 2xz + z^2 + 4yz^2}{4y} = \frac{(1+4y)z^2 - 2xz + x^2}{4y}$$

$$= \frac{(1+4y)\left[z^2 - 2z\left(\frac{x}{1+4y}\right) + \frac{x^2}{(1+4y)^2}\right] + x^2 - \frac{x^2}{1+4y}}{4y}$$

20 $$= \left(\frac{1+4y}{4y}\right)\left[z - \frac{x}{1+4y}\right]^2 + \frac{x^2}{1+4y}$$

$$\therefore\ u(x,y) = \frac{1}{\sqrt{4\pi y}} \int_{-\infty}^{\infty} e^{-\left(\frac{1+4y}{4y}\right)\left[z - \frac{x}{1+4y}\right]^2} \cdot e^{-\frac{x^2}{1+4y}}\, dz$$

Let $p = \sqrt{\frac{1+4y}{4y}}\left(z - \frac{x}{1+4y}\right)$

Then $dp = \sqrt{\frac{1+4y}{4y}}\, dz$

25 $$= \frac{1}{\sqrt{4\pi y}} \int_{-\infty}^{\infty} e^{-p^2} \cdot e^{-\frac{x^2}{1+4y}} \sqrt{\frac{4y}{1+4y}}\, dp$$

$$= \frac{e^{-\frac{x^2}{1+4y}}}{\sqrt{1+4y}} \cdot \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}\, dp.$$

28 $$= \boxed{\frac{e^{-\frac{x^2}{1+4y}}}{\sqrt{1+4y}}}$$

#5. $\mathcal{F}(u_{xx} + u_{yy})(\xi) = 0$

$$\frac{\partial^2}{\partial y^2}\mathcal{F}(u)(\xi) + (i\xi)^2\mathcal{F}(u)(\xi) = 0$$

$$\frac{\partial^2 \mathcal{F}(u)(\xi)}{\partial y^2} - \xi^2\mathcal{F}(u)(\xi) = 0$$

$$\therefore \mathcal{F}(u)(\xi) = c_1(\xi)e^{\xi y} + c_2(\xi)e^{-\xi y}$$

$$0 = \lim_{y\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,y)e^{-i\xi x}\,dx = \lim_{y\to\infty}\left[c_1(\xi)e^{\xi y} + c_2(\xi)e^{-\xi y}\right].$$

If $\xi > 0$, it follows that $c_1(\xi) = 0$; likewise, if $\xi < 0$ then $c_2(\xi) = 0$.

Thus $\mathcal{F}(u)(\xi) = \left\{\begin{array}{ll} c_2(\xi)e^{-\xi y} & \text{if } \xi > 0 \\ c(\xi)e^{\xi y} & \text{if } \xi < 0 \end{array}\right\} = c(\xi)e^{-|\xi| y}.$

Setting $y = 0$, $\mathcal{F}(\chi_{(-1,1)})(\xi) = \mathcal{F}(u(\cdot,0))(\xi) = c(\xi)$.

Also, Table Entry C with $a = y$ gives $\mathcal{F}\left(\sqrt{\frac{2}{\pi}}\frac{y}{(\cdot)^2+y^2}\right)(\xi) = e^{-y|\xi|}$. Thus

$$\mathcal{F}(u)(\xi) = \mathcal{F}(\chi_{(-1,1)})(\xi)\,\mathcal{F}\left(\sqrt{\frac{2}{\pi}}\frac{y}{(\cdot)^2+y^2}\right)(\xi) = \frac{1}{\sqrt{2\pi}}\mathcal{F}\left(\sqrt{\frac{2}{\pi}}\frac{y}{(\cdot)^2+y^2} * \chi_{(-1,1)}\right)(\xi)$$

$$\therefore u(x,y) = \frac{1}{\pi}\frac{y}{(\cdot)^2+y^2} * \chi_{(-1,1)}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-z)^2+y^2}\chi_{(-1,1)}(z)\,dz$$

$$= \frac{1}{\pi}\int_{-1}^{1}\frac{y}{(x-z)^2+y^2}\,dz = \frac{1}{\pi}\operatorname{Arctan}\left(\frac{z-x}{y}\right)\Big|_{z=-1}^{1}$$

$$\boxed{u(x,y) = \frac{1}{\pi}\left(\operatorname{Arctan}\left(\frac{1-x}{y}\right) + \operatorname{Arctan}\left(\frac{1+x}{y}\right)\right)}$$

#6 (a) Let $f, g \in V$. Then

$$\langle Tf, g\rangle = \int_0^1 -f''(x)\overline{g(x)}\,dx \overset{\text{Two integrations by parts}}{=} \Big[f(x)\overline{g'(x)} - f'(x)\overline{g(x)}\Big]\Big|_0^1 - \int_0^1 f(x)\overline{g''(x)}\,dx.$$

3 But $f(1) = 0 = \overline{g(1)}$ and $\overline{g'(0)} = 0 = f'(0)$ so $\langle Tf, g\rangle = \langle f, Tg\rangle$; i.e. T is hermitian on V.

(b) $\underset{\text{on } V}{Tf = \lambda f} \iff f''(x) + \lambda f(x) = 0,\ f'(0) = 0,\ f(1) = 0$. Since T is hermitian on V, all its eigenvalues are real. Since $-f(x)f'(x)\Big|_0^1 = 0$ for all real-valued f in V, all the eigenvalues of T on V are positive, say $\lambda = \beta^2$. Then $f(x) = A\cos(\beta x) + B\sin(\beta x)$ and $f'(x) = -\beta A\sin(\beta x) + \beta B\cos(\beta x)$.

$0 = f'(0) = \beta B \Rightarrow B = 0$. $0 = f(1) = A\cos(\beta) \Rightarrow \beta_n = (2n+1)\frac{\pi}{2} = (n+\frac{1}{2})\pi$ $(n = 0, 1, \ldots$

4 $\therefore$ $\boxed{\lambda_n = (n+\tfrac{1}{2})^2\pi^2 \text{ and } X_n(x) = \cos((n+\tfrac{1}{2})\pi x) \quad (n = 0, 1, 2, \ldots)}$

3 (c) Yes, $\{X_n\}_{n=0}^{\infty}$ is orthogonal on $(0,1)$ because they are eigenfunctions of a hermitian operator, corresponding to distinct eigenvalues.

6 (e) $\varphi(x) = 1 - x^2$, $\varphi'(x) = -2x$, $\varphi''(x) = -2$, are all continuous on $[0,1]$. $\varphi(1) \overset{\checkmark}{=} 0$ and $\varphi'(0) \overset{\checkmark}{=} 0$. By Theorem 2, the generalized Fourier series in part (d) converges uniformly to φ on $[0,1]$. Hence, the generalized Fourier series also converges in the mean square sense and in the pointwise sense to φ on $[0,1]$.

6 (d) $$A_n = \frac{\langle \varphi, X_n\rangle}{\langle X_n, X_n\rangle} = 2\int_0^1 \overbrace{(1-x^2)}^{U}\overbrace{\cos((n+\tfrac{1}{2})\pi x)\,dx}^{dV} = \cancel{\frac{2(1-x^2)\sin((n+\frac{1}{2})\pi x)}{(n+\frac{1}{2})\pi}\Big|_0^1}^{\,0} + \int_0^1 \frac{\overbrace{4x}^{U}\,\overbrace{\sin((n+\frac{1}{2})\pi x)}^{dV}}{(n+\frac{1}{2})\pi}$$

$$= \cancel{\frac{-4x\cos((n+\frac{1}{2})\pi x)}{(n+\frac{1}{2})^2\pi^2}\Big|_0^1}^{\,0} + \int_0^1 \frac{4\cos((n+\frac{1}{2})\pi x)}{(n+\frac{1}{2})^2\pi^2}\,dx = \frac{4\sin((n+\frac{1}{2})\pi x)}{(n+\frac{1}{2})^3\pi^3}\Big|_0^1 = \frac{4(-1)^n}{(n+\frac{1}{2})^3\pi^3} = \frac{32(-1)^n}{(2n+1)^3\pi^3}.$$

$$\therefore \boxed{\varphi(x) \sim \sum_{n=0}^{\infty} \frac{32(-1)^n\cos((n+\frac{1}{2})\pi x)}{(2n+1)^3\pi^3}}$$

#6 (f) We seek nontrivial solutions of the form $u(x,t) = X(x)T(t)$ to the homogeneous part of the problem; i.e. ①-②-③-④. This leads to

$$\begin{cases} X''(x) + \lambda X(x) = 0, & X'(0) = 0 = X(1), \\ T''(t) + \lambda T(t) = 0, & T'(0) = 0. \end{cases}$$

By part (b), the eigenvalues/eigenfunctions are $\lambda_n = (n+\frac{1}{2})^2\pi^2$ and $X_n(x) = \cos((n+\frac{1}{2})\pi x)$ $(n=0,1,2,\dots)$. The solution to the t-problem corresponding to $\lambda = \lambda_n$ is (up to a constant factor) $T_n(t) = \cos((n+\frac{1}{2})\pi t)$. Hence

$$u(x,t) = \sum_{n=0}^{\infty} c_n \cos((n+\tfrac{1}{2})\pi x)\cos((n+\tfrac{1}{2})\pi t)$$

is a formal solution to ①-②-③-④. We want to choose the coefficients so that ⑤ is met:

$$1 - x^2 = u(x,0) = \sum_{n=0}^{\infty} c_n \cos((n+\tfrac{1}{2})\pi x) \qquad \text{for all } 0 \le x \le 1.$$

By parts (d) and (e), $c_n = 32(-1)^n/(2n+1)^3\pi^3$ for $n=0,1,2,\dots$ Thus the solution is

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} \frac{32(-1)^n \cos((n+\frac{1}{2})\pi x)\cos((n+\frac{1}{2})\pi t)}{(2n+1)^3\pi^3}}$$

10 <u>Bonus</u>: The solution to part (f) is unique. This follows from energy methods. To see this let v be another solution, and consider the energy function

$$E(t) = \frac{1}{2}\int_0^1 \left[w_t^2(x,t) + w_x^2(x,t)\right] dx$$

of the difference $w(x,t) = u(x,t) - v(x,t)$. Note that w solves the problem ①-②-③-④ and ⑤′ $w(x,0) = 0$ for $0 \le x \le 1$. Then $dE/dt = \int_0^1 [w_t(x,t)w_{tt}(x,t) + w_x(x,t)w_x$

$$\overset{①}{=} \int_0^1 \left[w_t(x,t)w_{xx}(x,t) + w_x(x,t)w_{xt}(x,t)\right]dx = \int_0^1 \frac{\partial}{\partial x}\left[w_t(x,t)w_x(x,t)\right]dx = w_t(x,t)w_x(x,t)\Big|_{x=0}^{1} \overset{③-④}{=} 0.$$

Thus $E(t) = E(0) = \frac{1}{2}\int_0^1 \left[w_t^2(x,0) + w_x^2(x,0)\right]dx \overset{④-⑤'}{=} 0$ for all $t \ge 0$. By the vanishing theorem $w_t(x,t) = w_x(x,t) = 0$, and hence by ⑤′, $w(x,t) = 0$; i.e. $v = u$.

#7.

8 (a) $\begin{cases} \nabla^2 u = 0 & \text{in } 0 \le r < 1, \\ u(1;\theta) = \begin{cases} 50 & \text{if } 0<\theta<\pi, \\ -50 & \text{if } -\pi<\theta<0. \end{cases} \end{cases}$

16 Poisson: $u(r;\theta) = \dfrac{1-r^2}{2\pi}\displaystyle\int_{-\pi}^{\pi} \frac{h(\varphi)\,d\varphi}{1-2r\cos(\theta-\varphi)+r^2} = \frac{1-r^2}{2\pi}\left[\int_{-\pi}^{0} \frac{-50\,d\varphi}{1-2r\cos(\theta-\varphi)+r^2} + \int_{0}^{\pi} \frac{50\,d\varphi}{1-2r\cos(\theta-\varphi)+r^2}\right]$. Using the integral formula $\displaystyle\int \frac{d\psi}{A+B\cos(\psi)} = \frac{2}{\sqrt{A^2-B^2}}\,\mathrm{Arctan}\left(\tan(\psi/2)\,\frac{\sqrt{A^2-B^2}}{A+B}\right)$ if $A>|B|$, gives

$$u(r;\theta) = \frac{\cancel{1-r^2}}{\cancel{2}\pi}\cdot\frac{\cancel{2}(-50)}{\cancel{\sqrt{(1+r^2)^2-4r^2}}}\,\mathrm{Arctan}\left(\frac{\tan\left(\frac{\varphi-\theta}{2}\right)\sqrt{(1+r^2)^2-4r^2}}{1+r^2-2r}\right)\Bigg|_{\varphi=-\pi}^{0}$$

$$+\ \frac{\cancel{1-r^2}}{\cancel{2}\pi}\cdot\frac{\cancel{2}(50)}{\cancel{\sqrt{(1+r^2)^2-4r^2}}}\,\mathrm{Arctan}\left(\frac{\tan\left(\frac{\varphi-\theta}{2}\right)\sqrt{(1+r^2)^2-4r^2}}{1+r^2-2r}\right)\Bigg|_{\varphi=0}^{\pi}$$

$$= \frac{-50}{\pi}\,\mathrm{Arctan}\left(\tan\left(-\tfrac{\theta}{2}\right)\left(\frac{1+r}{1-r}\right)\right) + \frac{50}{\pi}\,\mathrm{Arctan}\left(\cot(\theta/2)\left(\frac{1+r}{1-r}\right)\right)$$

$$+\ \frac{50}{\pi}\,\mathrm{Arctan}\left(\cot(\theta/2)\left(\frac{1+r}{1-r}\right)\right) - \frac{50}{\pi}\,\mathrm{Arctan}\left(\tan\left(-\tfrac{\theta}{2}\right)\left(\frac{1+r}{1-r}\right)\right)$$

21 $$= \boxed{\frac{100}{\pi}\left[\mathrm{Arctan}\left(\cot(\theta/2)\left(\frac{1+r}{1-r}\right)\right) + \mathrm{Arctan}\left(\tan(\theta/2)\left(\frac{1+r}{1-r}\right)\right)\right]}$$

(b) Mean Value Property: $u(0;\theta) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} u(1;\varphi)\,d\varphi = \boxed{0}$

Alternate solution to #7(a)

$$u(r;\theta) = \sum_{n=-\infty}^{\infty} \hat{h}(n) e^{in\theta} \left(\frac{r}{a}\right)^{|n|} \quad \text{where } a=1 \text{ and}$$

$$\hat{h}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\varphi) e^{-in\varphi} d\varphi = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -50 e^{-in\varphi} d\varphi + \int_{0}^{\pi} 50 e^{-in\varphi} d\varphi\right]$$

$$= \frac{1}{2\pi}\left(\frac{-50 e^{-in\varphi}}{-in}\Bigg|_{-\pi}^{0}\right) + \frac{1}{2\pi}\left(\frac{50 e^{-in\varphi}}{-in}\Bigg|_{0}^{\pi}\right) \qquad (n \neq 0)$$

$$= \frac{50}{\pi} \cdot \frac{(1-(-1)^n)}{in} = \begin{cases} \dfrac{100}{\pi i (2k+1)} & \text{if } n = 2k+1 \text{ is odd,} \\ 0 & \text{if } n = 2k \text{ is even, } (n \neq 0) \end{cases}$$

$$\hat{h}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\varphi) d\varphi = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -50 d\varphi + \int_{0}^{\pi} 50 d\varphi\right] = 0.$$

$$\therefore \quad u(r;\theta) = \hat{h}(0) + \sum_{n=1}^{\infty} \left[\hat{h}(n) e^{in\theta} + \hat{h}(-n) e^{-in\theta}\right] r^n$$

$$= \sum_{k=0}^{\infty} \left[\frac{100 e^{i(2k+1)\theta}}{\pi i (2k+1)} - \frac{100 e^{-i(2k+1)\theta}}{\pi i (2k+1)}\right] r^{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{100\left(2i \sin((2k+1)\theta)\right) r^{2k+1}}{\pi i (2k+1)}$$

$$= \boxed{\frac{200}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\theta)\, r^{2k+1}}{2k+1}}$$

Final Exam Summer 2002 $(n=18)$

$\mu = 164.6$

$\sigma = 27.8$

Distribution of Scores

174 – 210	A	8
146 – 173	B	7
120 – 145	C	2
100 – 119	D	0
0 – 99	F	1