This is an open-textbook, open-notes test. That is, while solving the problems on this examination you may refer to your textbooks, Royden's *Real Analysis* and Karatzas and Shreve's *Brownian Motion and Stochastic Calculus*, or to the lecture notes you have taken for Math 416 this semester. Please solve any three of the following four problems and circle the number of each problem you want me to grade. Each problem is of equal value: 100 points. Thus, the total possible number of points on this examination is 300.

1. Let $E$ denote a Borel subset of a separable metric space $(X, d)$, let $m_\alpha(E)$ denote the Hausdorff $\alpha$-dimensional measure of $E$, and let $P$ denote the Cantor ternary set in the interval $[0, 1]$.

   (a) Show that $\inf \{ \alpha : m_\alpha(E) = 0 \} = \sup \{ \beta : m_\beta(E) = \infty \}$.

   (Note: The common value of the preceding infimum and supremum is called the Hausdorff dimension of the set $E$.)

   (b) If $\alpha > \log(2)/\log(3)$, show that $m_\alpha(P) = 0$.

   (c) If $\beta = \log(2)/\log(3)$, show that $m_\beta(P) = 1/2^\beta$.

   (d) What is the Hausdorff dimension of $P$?

2. Let $f_1, \ldots, f_n$ be measurable real-valued functions on a probability space $(X, A, \mu)$. Prove or disprove: $\mathcal{F} = \{ f_1, \ldots, f_n \}$ is an independent family if and only if the cumulative joint distribution function for the ordered $n$-tuple $(f_1, \ldots, f_n)$ satisfies

   $$F_{(f_1, \ldots, f_n)}(t_1, \ldots, t_n) = F_{f_1}(t_1) \cdots F_{f_n}(t_n)$$

   for all $(t_1, \ldots, t_n) \in \mathbb{R}^n$.

3. (a) Let $f \in L^2(0, 1)$ have the property that $\hat{f}(n) \neq 0$ for all integers $n$. Show that the set of all convolution products $f \ast g$, as $g$ ranges over $L^2(0, 1)$, is dense in $L^2(0, 1)$.

   (b) Let $f \in L^2(0, 1)$ have the property that the set of all convolution products $f \ast g$, as $g$ ranges over $L^2(0, 1)$, is dense in $L^2(0, 1)$. Show that $\hat{f}(n) \neq 0$ for all integers $n$.

4. (a) Show that there is no function $d \in L^1(\mathbb{R}^n)$ such that $d \ast f = f$ for all $f \in L^1(\mathbb{R}^n)$.

   (b) Show that there is a sequence of functions $\{d_k\}_{k=1}^\infty \subseteq L^1(\mathbb{R}^n)$ such that $d_k \ast f \to f$ in the $L^1(\mathbb{R}^n)$-norm for each $f \in L^1(\mathbb{R}^n)$.
The desired conclusion, \( \inf \{ \alpha : m^\alpha_E = 0 \} = \sup \{ \beta : m^\beta_E = \infty \} \), easily follows from (1) and (2) below.

(1) If \( m^{\alpha_0}_0(E) < \infty \) then \( m^\alpha_E = 0 \) for all \( \alpha > \alpha_0 \).

(2) If \( 0 < m^{\alpha_0}_0(E) \) then \( m^\beta_E = \infty \) for all \( \beta < \alpha_0 \).

**Proof of (1):** Suppose \( m^{\alpha_0}_0(E) < \infty \) and let \( \alpha > \alpha_0 \). Then to each sufficiently small \( \varepsilon > 0 \), there corresponds a covering of open balls
\[ \{ B_i \}_{i=1}^\infty \]
with each radius \( r_i \in (0, \varepsilon) \) and \( \sum_{i=1}^\infty r_i^{\alpha_0} < m^{\alpha_0}_0(E) + 1 \).

Then \( \sum_{i=1}^\infty r_i^{\alpha} = \sum_{i=1}^\infty r_i^{\alpha - \alpha_0} r_i^{\alpha_0} < \varepsilon^{\alpha - \alpha_0} \sum_{i=1}^\infty r_i^{\alpha_0} < \varepsilon^{\alpha - \alpha_0} (m^{\alpha_0}_0(E) + 1) \),

which tends to 0 as \( \varepsilon \to 0^+ \). Thus
\[
m^\alpha_E = \lim_{\varepsilon \to 0^+} \lambda^{(\varepsilon)}_\alpha(E) = \lim_{\varepsilon \to 0^+} \inf \{ \sum_{i=1}^\infty r_i^{\alpha} : E \subseteq \bigcup B_i, \text{ each } r_i \in (0, \varepsilon) \}
= 0.
\]

**Proof of (2):** Suppose \( 0 < m^{\alpha_0}_0(E) \) and let \( \beta < \alpha_0 \). Then for each sufficiently small \( \varepsilon > 0 \) and for every covering of \( E \) by open balls
\[ \{ B_i \}_{i=1}^\infty \]
with each radius \( r_i \in (0, \varepsilon) \), we have \( \sum_{i=1}^\infty r_i^{-\alpha_0} > m^{\alpha_0}_0(E) > 0 \).

Consequently,
\[
\sum_{i=1}^\infty r_i^{\beta} = \sum_{i=1}^\infty r_i^{\beta - \alpha_0} r_i^{\alpha_0} > \varepsilon^{\beta - \alpha_0} \sum_{i=1}^\infty r_i^{\alpha_0} > \varepsilon^{\beta - \alpha_0} \frac{m^{\alpha_0}_0(E)}{2}
\]
which tends to \( \infty \) as \( \varepsilon \to 0^+ \). Therefore
\[
m^\beta_E = \lim_{\varepsilon \to 0^+} \lambda^{(\varepsilon)}_\beta(E) = \lim_{\varepsilon \to 0^+} \inf \{ \sum_{i=1}^\infty r_i^{\beta} : E \subseteq \bigcup B_i, \text{ each } r_i \in (0, \varepsilon) \} = \infty.
\]
#1. (b) Suppose $\alpha > \frac{\log(2)}{\log(3)}$; i.e. suppose $3^\alpha > 2$. Given $\epsilon > 0$, choose $k_0 \in \mathbb{N}$ such that $\frac{1}{3^{k_0}} < \epsilon$. For all $k \geq k_0$, $2^k$ closed disjoint intervals of length $\frac{1}{3^k}$ cover $\mathcal{P}$. The interiors of these intervals cover all but a finite number of points of $\mathcal{P}$. Therefore

$$\lambda_\alpha^{(\epsilon)}(\mathcal{P}) \leq \sum_{i=1}^{2^k} r_i^\alpha = \sum_{i=1}^{2^k} \left( \frac{1}{2 \cdot 3^k} \right)^\alpha = 2^k \left( \frac{1}{2 \cdot 3^k} \right)^\alpha = \frac{1}{2^k} \left( \frac{2}{3^k} \right)^\alpha \rightarrow 0$$

as $k \to \infty$. Thus $\lambda_\alpha^{(\epsilon)}(\mathcal{P}) = 0$ for all $\epsilon > 0$ so $m_\alpha(\mathcal{P}) = \lim_{\epsilon \to 0^+} \lambda_\alpha^{(\epsilon)}(\mathcal{P}) = 0$.

(c) Suppose $\beta = \frac{\log(2)}{\log(3)}$; i.e. suppose $3^\beta = 2$. Let $\epsilon > 0$ and let $\{B_i \}_{i=1}^\infty$ be a sequence of open balls (i.e. open intervals) in $[0,1]$ such that $\mathcal{P} \subseteq \bigcup_{i=1}^\infty B_i$ and $r_i = \text{radius of } B_i \in (0, \epsilon)$ for all $i \geq 1$. Since $\mathcal{P}$ is compact there exists a finite subcover $B_{i_1}, \ldots, B_{i_n}$ of $\mathcal{P}$. Recall that $\mathcal{P} = \bigcap_{k=1}^\infty \mathcal{P}_k$ where each $\mathcal{P}_k$ consists of the union of $2^k$ closed intervals of length $\frac{1}{3^k}$ obtained at the $k$th stage in the construction of the Cantor set. Let $\delta > 0$ and let $(1+\delta)B_{i_j}$ denote the dilate by $1+\delta$ of the ball $B_{i_j}$ about its center (i.e. midpoint). Choose $k \in \mathbb{N}$ sufficiently large so $\frac{1}{3^k} < \delta \cdot \min\{ r_{i_j} : 1 \leq j \leq n \}$. It follows that $\frac{1}{3^k} + r_{i_j} < (1+\delta) r_{i_j}$ for all $1 \leq j \leq n$, so if one of the closed intervals $I$ of length $\frac{1}{3^k}$ in the collection of $2^k$ intervals generating $\mathcal{P}_k$ has nonempty intersection with some ball $B_{i_j}$, then $I \subseteq (1+\delta)B_{i_j}$. It follows that $\mathcal{P}_k \subseteq \bigcup_{j=1}^n (1+\delta)B_{i_j}$ so

$$\frac{1}{2^k} = 2^k \left( \frac{1}{2 \cdot 3^k} \right)^\beta = \sum_{i=1}^{2^k} r_i^\beta \leq \sum_{j=1}^n [(1+\delta) r_{i_j}]^\beta \leq (1+\delta)^\beta \sum_{i=1}^\infty r_i^\beta,$$

where $r_{i_j}$ is the radius of the $i$th interval (ball) in the collection generating $\mathcal{P}_k$. 

\[ \text{radius of } B_{i_j} \]
Since $\delta > 0$ is arbitrary, \( \frac{1}{2^B} \leq \sum_{i=1}^{\infty} r_i^B \). Hence

\[
\frac{1}{2^B} \leq \inf \left\{ \sum_{i=1}^{\infty} r_i^B : P \leq \bigcup_{i=1}^{\infty} B_i, \text{ each } r_i < \varepsilon \right\} = \lambda^B_\varepsilon(P).
\]

Since $\varepsilon > 0$ is arbitrary, \( \frac{1}{2^B} \leq \lim_{\varepsilon \to 0^+} \lambda^B_\varepsilon(P) = m_\beta(P) \). On the other hand, the argument in part (a) shows that if $\frac{1}{3}k < \varepsilon$ then

\[
\lambda^B_\varepsilon(P) \leq \sum_{i=1}^{2^k} r_i^B = \sum_{i=1}^{2^k} (\frac{1}{2} \cdot \frac{1}{3^k})^B = \frac{1}{2^B} \left( \frac{2}{3^B} \right)^k = \frac{1}{2^B}.
\]

Hence $m_\beta(P) = \frac{1}{2^B}$.

(d) \( \dim_{\text{Haus}}(P) = \frac{\log(2)}{\log(3)} \).
#2. Let \( F = \{f_1, \ldots, f_n\} \) be a set of \( n \) measurable, real-valued functions on a probability space \((\mathbb{X}, \mathcal{A}, \mu)\).

(\Rightarrow) Suppose that \( F \) is an independent family. If \((t_1, \ldots, t_n) \in \mathbb{R}^n\), then
\[
F_{(f_1, \ldots, f_n)}(t_1, \ldots, t_n) = \mu \left( \bigcap_{i=1}^{n} \{ x \in \mathbb{X} : f_i(x) \leq t_i \} \right)
= \prod_{i=1}^{n} \mu \left( \{ x \in \mathbb{X} : f_i(x) \leq t_i \} \right) \quad \text{(by independence)}
= \prod_{i=1}^{n} F_i(t_i).
\]

(\Leftarrow) Suppose that the cumulative joint distribution function for \((f_1, \ldots, f_n)\) satisfies
\[
F_{(f_1, \ldots, f_n)}(t_1, \ldots, t_n) = \prod_{i=1}^{n} F_i(t_i) \quad \text{for all } (t_1, \ldots, t_n) \in \mathbb{R}^n.
\]

Define the (Borel-)measurable transformation \( T : \mathbb{X} \to \mathbb{R}^n \) by
\[
T(x) = (f_1(x), \ldots, f_n(x)) \quad (x \in \mathbb{X}).
\]

Let \((t_1, \ldots, t_n) \in \mathbb{R}^n\) and \( M_i = (-\infty, t_i] \) for \( 1 \leq i \leq n \). Then
\[
(\mu T^{-1})(M_1 \times \ldots \times M_n) = \mu \left( T^{-1}[M_1 \times \ldots \times M_n] \right)
= \mu \left( \{ x \in \mathbb{X} : T(x) \in M_1 \times \ldots \times M_n \} \right)
= \mu \left( \bigcap_{i=1}^{n} \{ x \in \mathbb{X} : f_i(x) \leq t_i \} \right)
= F_{(f_1, \ldots, f_n)}(t_1, \ldots, t_n).
\]

On the other hand,
\[
(\mu f_1^{-1} \times \ldots \times \mu f_n^{-1})(M_1 \times \ldots \times M_n) = (\mu f_1^{-1})(M_1) \cdot \ldots \cdot (\mu f_n^{-1})(M_n)
\]
\[
= \prod_{i=1}^{n} \mu(\{x \in X : f_i(x) \leq t_i\})
\]
\[
= \prod_{i=1}^{n} F_i(t_i).
\]

It follows then from (\ast) that \(\mu T^{-1}\) and \(\mu f_1^{-1} \times \ldots \times \mu f_n^{-1}\) are Borel measures on \(\mathbb{R}\) which agree on the class of subsets of the form \(I_1 \times \ldots \times I_n\) where each \(I_i = (-\infty, b_i]\) for some \(b_i \in \mathbb{R}\). Let \(\mathcal{C}\) denote the class of intervals of \(\mathbb{R}\) of the form \(I = (-\infty, b]\) for some \(b \in \mathbb{R}\). It is easy to see that \(\mathcal{C}\) is a \(\pi\)-system:

\[
(-\infty, b_1] \cap (-\infty, b_2] = (-\infty, b_1 \wedge b_2] \in \mathcal{C}.
\]

Let \(\mathcal{D}\) denote the collection of Borel sets \(B \in \mathcal{B}\) such that for any sets from \(\mathcal{C}\), say \(B_1, \ldots, B_n\), we have the equality

\[
\mu T^{-1}(B_1 \times \ldots \times B_n) = (\mu f_1^{-1} \times \ldots \times \mu f_n^{-1})(B_1 \times \ldots \times B_n)
\]

Claim: \(\mathcal{D}\) is a \(\lambda\)-system.

Remark: Clearly \(\mathcal{C} \subseteq \mathcal{D}\). It will then follow from the claim and Dynkin's \(\pi\)-\(\lambda\) theorem that \(\mathcal{D}\) contains the \(\sigma\)-algebra generated by \(\mathcal{C}\). But clearly \(\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})\), so it will follow that \(\mathcal{B}(\mathbb{R}) \subseteq \mathcal{D}\). That is, for all Borel sets \(B_1, \ldots, B_n\) in \(\mathbb{R}\) we have
\[
\mu \left( \bigcap_{i=1}^{n} \{ x \in \mathbb{X}: f_i(x) \in B_i \} \right) = \mu_T^{-1}(B_1 \times \ldots \times B_n) \\
= (\mu f_1^{-1} \times \ldots \times \mu f_n^{-1})(B_1 \times \ldots \times B_n) \\
= \prod_{i=1}^{n} \mu \left( \{ x \in \mathbb{X}: f_i(x) \in B_i \} \right),
\]
and thus \( f = \{ f_1, \ldots, f_n \} \) is an independent family.

Proof of Claim:

(i') Let \( B_1 = IR \) and \( B_2, \ldots, B_n \in \mathcal{D} \). Then for each \( b \in IR \), \(-\infty, b \) \( \in \mathcal{D} \)

\[
\mu_T^{-1}(\mathcal{D}, b) = \bigcap_{i=1}^{n} \{ x \in \mathbb{X}: f_i(x) \in B_i \} \\
= (\mu f_1^{-1} \times \ldots \times \mu f_n^{-1})(\mathcal{D}, b)
\]

Hence

\[
\mu_T^{-1}(IR \times B_2 \times \ldots \times B_n) = \lim_{m \to \infty} \mu_T^{-1}(\mathcal{D}, m) = (\mu f_1^{-1} \times \ldots \times \mu f_n^{-1})(IR \times B_2 \times \ldots \times B_n)
\]

Since \( B_2, \ldots, B_n \in \mathcal{D} \) were arbitrary, \( B_1 = IR \in \mathcal{D} \), by definition.

(ii') Suppose \( B_1 \in \mathcal{D} \) and let \( B_2, \ldots, B_n \in \mathcal{D} \). Then

\[
\mu_T^{-1}(IR \times B_1 \times B_2 \times \ldots \times B_n) = \mu_T^{-1}(IR \times B_2 \times \ldots \times B_n) - \mu_T^{-1}(B_1 \times B_2 \times \ldots \times B_n)
\]

\[
= (\mu f_1^{-1} \times \ldots \times \mu f_n^{-1})(IR \times B_2 \times \ldots \times B_n) - (\mu f_1^{-1} \times \ldots \mu f_n^{-1})(B_1 \times B_2 \times \ldots \times B_n)
\]

Consequently \( IR \times B_1 \in \mathcal{D} \).
(iii') Let \( \{ A_k \}_{k=1}^{\infty} \subseteq \mathcal{D} \) with \( A_k \cap A_l = \emptyset \) if \( k \neq l \), and let \( B_2, \ldots, B_n \in \mathcal{D} \). Then

\[
\mu T^{-1} \left( \bigcup_{k=1}^{\infty} A_k \times B_2 \times \ldots \times B_n \right) = \lim_{K \to \infty} \mu T^{-1} \left( \bigcup_{k=1}^{K} A_k \times B_2 \times \ldots \times B_n \right)
\]

\[
= \lim_{K \to \infty} \bigcup_{k=1}^{K} \left( \mu T^{-1} \left( A_k \times B_2 \times \ldots \times B_n \right) \right)
\]

\[
= \lim_{K \to \infty} \bigcup_{k=1}^{K} \left( \mu f_1^{-1} \times \ldots \times \mu f_n^{-1} \right) \left( A_k \times B_2 \times \ldots \times B_n \right)
\]

\[
= \lim_{K \to \infty} \left( \mu f_1^{-1} \times \ldots \times \mu f_n^{-1} \right) \left( \bigcup_{k=1}^{K} A_k \times B_2 \times \ldots \times B_n \right)
\]

\[
= \left( \mu f_1^{-1} \times \ldots \times \mu f_n^{-1} \right) \left( \bigcup_{k=1}^{\infty} A_k \times B_2 \times \ldots \times B_n \right)
\]

Since \( B_2, \ldots, B_n \in \mathcal{D} \) were arbitrary, the definition of \( \mathcal{D} \) implies \( \bigcup_{k=1}^{\infty} A_k \in \mathcal{D} \). Therefore \( \mathcal{D} \) is a \( \lambda \)-system. This concludes the proof of the claim and hence the solution of problem 2.
#3. Let \( f \in L^2(0,1) \).

(a) Suppose that \( \hat{f}(n) \neq 0 \) for all \( n \in \mathbb{Z} \). Let \( h \in L^2(0,1) \) and let \( N \) be a positive integer. Then denoting \( e_n(t) = e^{2\pi int} \) for \( n \in \mathbb{Z} \), \( t \in (0,1) \), we have

\[
S_N(h; t) = \sum_{k=-N}^{N} \hat{h}(k)e_k(t) = \sum_{k=-N}^{N} \frac{\hat{h}(k)}{\hat{f}(k)} \hat{f}(k)e_k(t) = \sum_{k=-N}^{N} \hat{h}(k)(f* e_k)(t)
\]

= \( f* \left( \sum_{k=-N}^{N} \frac{\hat{h}(k)}{\hat{f}(k)} e_k(t) \right) \). Note that \( g_N(t) = \sum_{k=-N}^{N} \frac{\hat{h}(k)}{\hat{f}(k)} e_k(t) \)

is a trigonometric polynomial and hence belongs to \( L^2(0,1) \). Also

\[
S_N(h; t) = (f*g_N)(t) \quad \text{for all} \quad t \in (0,1) \quad \text{and} \quad \| S_N(h) - h \| \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty.
\]

(This is a consequence of Parseval's identity; see the Math 315 lecture notes on the \( L^2 \)-theory of Fourier series or Rudin's Principles of Mathematical Analysis, Theorems 8.16 and 11.40.) It follows that the set of all convolution products \( f*g \), as \( g \) ranges over \( L^2(0,1) \), is dense in \( L^2(0,1) \).

(b) Suppose that the set of all convolution products \( f*g \), as \( g \) ranges over \( L^2(0,1) \), is dense in \( L^2(0,1) \). Suppose \( \hat{f}(n_0) = 0 \) for some \( n_0 \in \mathbb{Z} \). Then \( \hat{f} \hat{g}(n_0) = \hat{f}(n_0) \hat{g}(n_0) = 0 \) for all \( g \in L^2(0,1) \). For all \( g \in L^2(0,1) \), we have by Parseval's identity

\[
\| e_{n_0} - f*g \|_{L^2(0,1)}^2 = 1 + \sum_{n \neq n_0} |1 - \hat{f} \hat{g}(n)|^2 \geq 1,
\]

so the set \( f*g \), as \( g \) ranges over \( L^2(0,1) \), is not dense in \( L^2(0,1) \), a contra-
#4. (a) Suppose that there exists a function $d \in L^1(\mathbb{R}^n)$ such that $d * f = f$ for all $f \in L^1(\mathbb{R}^n)$. In particular, $d * d = d$.

If $\xi \in \mathbb{R}^n$ and $f \in L^1(\mathbb{R}^n)$, the Fourier transform of $f$ at $\xi$ is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(\eta) e^{-i\eta \cdot \xi} d\eta$. It is easy to show that $(f * g)(\xi) = \hat{f}(\xi) \hat{g}(\xi)$ for all $f, g \in L^1(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$. Hence $d * d = d$ implies $(\hat{d}(\xi))^2 = (\hat{d} + \hat{d})(\xi) = \hat{d}(\xi)$ for all $\xi \in \mathbb{R}^n$ and hence, for each $\xi \in \mathbb{R}^n$, $\hat{d}(\xi)$ is either 0 or 1. But $\xi \mapsto \hat{d}(\xi)$ is a continuous function on $\mathbb{R}^n$ and satisfies $|\hat{d}(\xi)| \to 0$ as $||\xi|| \to \infty$ when $f \in L^1(\mathbb{R}^n)$. It follows that $\hat{d}(\xi) = 0$ for all $\xi \in \mathbb{R}^n$ or $\hat{d}(\xi) = 1$ for all $\xi \in \mathbb{R}^n$, and clearly the second alternative is impossible since $|\hat{d}(\xi)| \to 0$ as $||\xi|| \to \infty$. But then $\hat{d}(\xi) = 0$ for all $\xi \in \mathbb{R}^n$ implies $d(x) = 0$ a.e. by the uniqueness theorem, so $f(x) = (d * f)(x) = 0$ a.e. for all $f \in L^1(\mathbb{R}^n)$, a clear absurdity.

Therefore, there is no $d \in L^1(\mathbb{R}^n)$ such that $d * f = f$ for all $f \in L^1(\mathbb{R}^n)$.

(b) For $R > 0$ and $\Theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$, define the Fejer kernel by

$$K_R(\Theta) = \frac{1}{(2\pi R)^n} \int_0^R \cdots \int_0^R \left( \int_0^{e^{i(\theta_1 \cdot \xi_1) + \cdots + \theta_n \cdot \xi_n}} \right) (d\xi_1 \cdots d\xi_n) d\eta_1 \cdots d\eta_n$$

This kernel can be motivated by looking at the $C$-means of the partial Fourier
integrals of \( f \in L^1(\mathbb{R}^n) \):  
\[
\sigma(f; x) = \frac{1}{R^n} \int_0^R \cdots \int_0^R S_{(-p_0 \cdots -p_n)}(f; x) \, dp_1 \cdots dp_n
\]
\[
= \int_{\mathbb{R}^n} f(y) K_R(x-y) \, dy = (f * K_R)(x).
\]
One proves the identities
\[
K_R(\theta) = \frac{1}{(2\pi R^n)} \int_0^R \cdots \int_0^R \frac{2 \sin(p_i \theta_i)}{\theta_i} \cdots \frac{2 \sin(p_n \theta_n)}{\theta_n} \, dp_1 \cdots dp_n
\]
\[
= \left( \frac{2 \sin^2(R \theta_i/2)}{\pi R \theta_i^2} \right) \cdots \left( \frac{2 \sin^2(R \theta_n/2)}{\pi R \theta_n^2} \right).
\]
and the properties below follow as in the \( n=1 \) dimensional case (see HW set #3, problem E):

1. \( \int_{\mathbb{R}^n} K_R(\theta) \, d\theta = 1 \) for all \( R > 0 \).

2. \( K_R(\theta) \geq 0 \) for all \( R > 0 \) and all \( \theta \in \mathbb{R}^n \).

3. \( \lim_{R \to \infty} \int_{\|\theta\| > \delta} \theta \) for every \( \delta > 0 \).

Claim: The sequence of functions \( \{K_k\}_{k=1}^\infty \subseteq L^1(\mathbb{R}^n) \) has
the property that \( K_k * f \to f \) in the \( L^1(\mathbb{R}^n) \)-norm for
each \( f \in L^1(\mathbb{R}^n) \).
Proof of Claim: Let \( f \in L^1(\mathbb{R}^n) \) and let \( \varepsilon > 0 \). Denote \( f_y(x) = f(x - y) \) for \( x, y \in \mathbb{R}^n \) and observe that \( \| f - f_y \|_{L^1(\mathbb{R}^n)} \to 0 \) as \( y \to 0 \). (This is clear if \( f \) is a continuous function of compact support on \( \mathbb{R}^n \), and the density of such functions in \( L^1(\mathbb{R}^n) \) yields the result for general \( f \in L^1(\mathbb{R}^n) \).) Therefore there is a \( \delta > 0 \) such that \( \| f - f_y \|_{L^1(\mathbb{R}^n)} < \frac{\varepsilon}{2} \) for all \( \| y \| < \delta \). Choose \( R_0 > 0 \) such that \( 2 \| f \|_{L^1(\mathbb{R}^n)} \int \frac{R}{R^2} R(\theta) d\theta < \frac{\varepsilon}{2} \) for all \( R \geq R_0 \) (cf. property (3) of the Fejer kernel). For \( R \geq R_0 \) we have

\[
\int_{\mathbb{R}^n} \left| K_R \ast f(x) - f(x) \right| dx = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} K_R(y) f(x-y) dy - \int_{\mathbb{R}^n} K_R(y) f(x) dy \right) dx
\]

\[
= \int_{\mathbb{R}^n} \left( \int_{\|y\| < \delta} K_R(y) \left| f(x-y) - f(x) \right| dy \right) dx + \int_{\mathbb{R}^n} \left( \int_{\|y\| \geq \delta} K_R(y) \left| f(x-y) - f(x) \right| dy \right) dx
\]

\[
\leq \int_{\|y\| < \delta} K_R(y) \| f - f_y \|_{L^1(\mathbb{R}^n)} dy + \int_{\|y\| \geq \delta} K_R(y) 2\| f \|_{L^1(\mathbb{R}^n)} dy
\]
\[
\leq \frac{\varepsilon}{2} \int_{\|y\|<s} K_k(y) \, dy + 2\|f\|_{L^1(\mathbb{R}^n)} \int_{\|y\|\geq s} K_k(y) \, dy \\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
= \varepsilon . \quad \text{Q.E.D.}
\]