Mathematics 325 Final Exam Summer 2004

Name: Dr. Grow (2 pts.)

1.(33 pts.) Consider a function of the form

$$u(x,y) = Ax^3 + Byx^2 + Cxy^2 + Dy^3$$

where $A$, $B$, $C$, and $D$ are constants.

(a) Find the most general function of the above form that solves $u_{xx} + u_{yy} = 0$ in the $xy$-plane.

(b) Find a solution of $u_{xx} + u_{yy} = 0$ in the $xy$-plane that satisfies $u(x,0) = 2x^3$ and $u_y(x,0) = 3x^2$ for all $-\infty < x < \infty$.

10 pts. (a) $u_x = 3Ax^2 + 2Byx + Cy^2$ and $u_y = Bx^2 + 2Cxy + 3Dy^2$.

12 $\therefore\ 0 = u_{xx} + u_{yy} = 6Ax + 2By + 2Cx + 6Dy = (6A+2C)x + (2B+6D)y$

8 $\therefore\ 6A+2C = 0$ and $2B+6D = 0$ so $C = -3A$ and $B = -3D$.

3 $\therefore\ \boxed{u(x,y) = A(x^3 - 3xy^2) + D(y^3 - 3yx^2)}$ or $Ax^3 - 3Dyx^2 - 3Axy^2 + Dy^3$

10 pts. (b) Assume a solution of the form in part (a). Then $u_y = -6Axy + D(3y^2 - 3x^2)$

4 $2x^3 = u(x,0) = Ax^3 \Rightarrow A = 2$

4 $3x^2 = u_y(x,0) = -3Dx^2 \Rightarrow D = -1$

2 $\therefore\ \boxed{u(x,y) = 2(x^3 - 3xy^2) - (y^3 - 3yx^2) = 2x^3 + 3x^2y - 6xy^2 - y^3}$

2.(33 pts.) Consider the partial differential equation

(*) $$u_{xx} + u_{yy} - 2u_{xy} + u_x - u_y = 0.$$

(a) Classify the order and type (linear, nonlinear, homogeneous, inhomogeneous, elliptic, parabolic, hyperbolic) of (*).

(b) Find, if possible, the general solution of (*) in the $xy$ – plane.

5 pts. (a) $\boxed{\text{2nd order, linear, homogeneous, parabolic}}$

$$B^2 - 4AC = (-2)^2 - 4(1)(1) = 0$$

26 pts. (b) (*) $$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0$$

4 $$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 u + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) u = 0$$

4 Let $\xi = x + y$ and $\eta = x - y$. Then the chain rule implies $\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}$ so $\frac{\partial}{\partial x} - \frac{\partial}{\partial y} = 2\frac{\partial}{\partial \eta}$. Thus (*)

4 becomes $4\frac{\partial^2 u}{\partial \eta^2} + 2\frac{\partial u}{\partial \eta} = 0 \Rightarrow \frac{\partial v}{\partial \eta} + \frac{1}{2} v = 0$ where $v = \frac{\partial u}{\partial \eta}$.

4 + 4 $\therefore$ $v = c_1(\xi) e^{-\eta/2}$ so $\frac{\partial u}{\partial \eta} = c_1(\xi) e^{-\eta/2} \Rightarrow u = c_3(\xi) e^{-\eta/2} + c_2(\xi)$.

6 That is, $\boxed{u(x,y) = f(x+y) e^{\frac{y-x}{2}} + g(x+y)}$ where $f$ and $g$ are arbitrary $C^2$-functions of a single real variable.

16 pts.

3.(33 pts.) (a) Use Fourier transform methods to derive the formula

$$u(x,t)=\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-y)^2}{4kt}}\phi(y)dy}{\sqrt{4k\pi t}}$$

for the solution to

$$u_t-ku_{xx}=0 \quad \text{in} \quad -\infty<x<\infty \quad \text{and} \quad 0<t<\infty$$

which satisfies the initial condition

$$u(x,0)=\phi(x) \quad \text{for} \quad -\infty<x<\infty.$$

17 pts. (b) Solve $u_t-u_{xx}=0$ in $-\infty<x<\infty$ and $0<t<\infty$, given that $u(x,0)=e^{-x}$ for $-\infty<x<\infty$.

(a) $\mathcal{F}(u_t-ku_{xx})(\xi)=\mathcal{F}(0)(\xi)$

2 $\frac{\partial}{\partial t}\mathcal{F}(u)(\xi)+k\xi^2\mathcal{F}(u)(\xi)=0$

1 $\mu=e^{\int k\xi^2dt}=e^{k\xi^2t}$

$e^{k\xi^2t}\frac{\partial}{\partial t}\mathcal{F}(u)(\xi)+k\xi^2e^{k\xi^2t}\mathcal{F}(u)(\xi)=0$

$\frac{\partial}{\partial t}\left(e^{k\xi^2t}\mathcal{F}(u)(\xi)\right)=0$

$e^{k\xi^2t}\mathcal{F}(u)(\xi)=c_1(\xi)$

2 $\mathcal{F}(u)(\xi)=c_1(\xi)e^{-k\xi^2t}$

2 $\mathcal{F}(\varphi)(\xi)=\mathcal{F}(u)(\xi)\Big|_{t=0}=c_1(\xi)$

By table entry I with $\frac{1}{4a}=kt$,

(i.e. $a=\frac{1}{4kt}$) $\mathcal{F}\left(\sqrt{2a}\,e^{-a(\cdot)^2}\right)(\xi)=e^{-\frac{\xi^2}{4a}}=e^{-\xi^2kt}$.

2 $\Rightarrow \mathcal{F}\left(\frac{1}{\sqrt{2kt}}e^{-\frac{(\cdot)^2}{4kt}}\right)(\xi)=e^{-\xi^2kt}$.

1 $\therefore \mathcal{F}(u)(\xi)=\mathcal{F}(\varphi)(\xi)\,\mathcal{F}\left(\frac{1}{\sqrt{2kt}}e^{-\frac{(\cdot)^2}{4kt}}\right)(\xi)$

2 $=\frac{1}{\sqrt{2\pi}}\mathcal{F}\left(\frac{\varphi*e^{-\frac{(\cdot)^2}{4kt}}}{\sqrt{2kt}}\right)(\xi)$

$\mathcal{F}\left(\varphi * e^{-\frac{(\cdot)^2}{4kt}}\right)(\xi)$

$\therefore u(x,t)=\frac{\varphi*e^{-\frac{(\cdot)^2}{4kt}}}{\sqrt{4k\pi t}}(x)$

$=\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-y)^2}{4kt}}\varphi(y)dy}{\sqrt{4k\pi t}}$ 2

(b) $u(x,t)=\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-y)^2}{4t}}\cdot e^{-y}}{\sqrt{4\pi t}}dy$ 2

$=\int_{-\infty}^{\infty}\frac{e^{-\frac{[(x-y)^2+4ty]}{4t}}}{\sqrt{4\pi t}}dy$. We complete the square 2

on $y$ in the exponent:

$-\left[\frac{(x-y)^2+4ty}{4t}\right]=-\left[\frac{x^2-2xy+y^2+4ty}{4t}\right]$

$=-\left[\frac{y^2+2y(2t-x)+(2t-x)^2-(2t-x)^2+\ldots}{4t}\right]$ 2

$=\frac{-(y+2t-x)^2+4t^2-4tx}{4t}$

$=-\frac{(y+2t-x)^2}{4t}+t-x$. Thus 1

$u(x,t)=\int_{-\infty}^{\infty}\frac{e^{-\frac{(y+2t-x)^2}{4t}}\cdot e^{t-x}}{\sqrt{4\pi t}}dy$.

Let $p=\frac{y+2t-x}{\sqrt{4t}}$. Then $dp=\frac{dy}{\sqrt{4t}}$ so

$u(x,t)=\frac{e^{t-x}}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-p^2}dp=\boxed{e^{t-x}}$.

4.(33 pts.) Solve $u_{tt}-u_{xx}=0$ ① for $0<x<\pi$ and $0<t<\infty$, with the boundary conditions $u(0,t)=0$ ② and $u(\pi,t)=0$ ③ for $t\ge 0$, and the initial conditions $u(x,0)=x(\pi-x)$ ⑤ and $u_t(x,0)=0$ ④ for $0\le x\le \pi$.

We seek nontrivial solutions to ①-②-③-④ of the form $u(x,t)=X(x)T(t)$.

① $\Rightarrow$ $T''(t)X(x) - X''(x)T(t)=0$ so $\dfrac{T''(t)}{T(t)} = \dfrac{X''(x)}{X(x)} = -\lambda$ (constant).

② $\Rightarrow$ $X(0)T(t)=0$, ③ $\Rightarrow$ $X(\pi)T(t)=0$, and ④ $\Rightarrow$ $X(x)T'(0)=0$. Thus

6 $$\left.\begin{cases} X''(x)+\lambda X(x)=0, \quad X(0)=0=X(\pi) \longrightarrow & \text{Eigenvalues: } \lambda_n = n^2 \\ T''(t)+\lambda T(t)=0, \quad T'(0)=0 & \text{Eigenfunctions: } X_n(x)=\sin(nx) \end{cases}\right\} n=1,2,3,\ldots$$

$\therefore$ $T_n(t)=A\cos(nt)+B\sin(nt)$ with $0=T_n'(0) = -nA\sin(nt)+nB\cos(nt)\Big|_{t=0} = nB \Rightarrow B=0.$

3 $\therefore$ $T_n(t)=\cos(nt)$ (up to a constant factor). Thus, a formal solution of ①-②-③-④ is

3 $$u(x,t)=\sum_{n=1}^{\infty} b_n\sin(nx)\cos(nt).$$ We need to choose the constants $b_1, b_2, \ldots$ so ⑤

3 is satisfied: $x(\pi-x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx)$ for all $0\le x\le\pi$. Therefore

3 $$b_n = \frac{\langle x(\pi-x), \sin(nx)\rangle}{\langle \sin(nx), \sin(nx)\rangle} = \frac{2}{\pi}\int_0^{\pi} \underbrace{x(\pi-x)}_{v}\underbrace{\sin(nx)dx}_{dv} = \frac{2}{\pi}\cancelto{0}{x(\pi-x)\left(-\frac{\cos(nx)}{n}\right)}\Bigg|_0^{\pi} -$$

$$\frac{2}{\pi}\int_0^{\pi}(\pi-2x)\left(-\frac{\cos(nx)}{n}\right)dx = \frac{2}{n\pi}\int_0^{\pi}\underbrace{(\pi-2x)}_{v}\underbrace{\cos(nx)dx}_{dv} = \frac{2}{n\pi}\cancelto{0}{(\pi-2x)\frac{\sin(nx)}{n}}\Bigg|_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi}\frac{\sin(nx)}{n}\Big($$

6 $$= \frac{4}{\pi n^2}\cdot\left(\frac{-\cos(nx)}{n}\right)\Bigg|_0^{\pi} = \frac{4(1-(-1)^n)}{\pi n^3} = \begin{cases} 0 & \text{if } n=2k \text{ is even,} \\ \dfrac{8}{\pi(2k+1)^3} & \text{if } n=2k+1 \text{ is odd.}\end{cases}$$

3 $$\therefore \boxed{u(x,t) = \sum_{k=0}^{\infty} \frac{8\sin((2k+1)x)\cos((2k+1)t)}{\pi(2k+1)^3}}.$$

5.(33 pts.) Solve the problem of heat conduction on a square:

$$u_t - u_{xx} - u_{yy} \overset{①}{=} 0 \quad \text{for } 0 < x < \pi,\ 0 < y < \pi,\ 0 < t < \infty,$$

given that for all times $t \geq 0$ the temperature $u = u(x, y, t)$ satisfies homogeneous Neumann boundary conditions on the four edges of the square:

$$\frac{\partial u}{\partial n} = 0 \quad \text{for } x = 0,\ x = \pi,\ y = 0, \text{ and } y = \pi,$$

and satisfies the initial condition

$$u(x, y, 0) \overset{⑥}{=} \cos^2(x)\cos^2(y) \quad \text{for } 0 \leq x \leq \pi,\ 0 \leq y \leq \pi.$$

The homogeneous Neumann boundary conditions are explicitly:

$$u_x(0,y,t) \overset{②}{=} 0 \quad \text{and} \quad u_x(\pi,y,t) \overset{③}{=} 0 \quad \text{for } 0 \le y \le \pi \text{ and } t \ge 0,$$

3

$$u_y(x,0,t) \overset{④}{=} 0 \quad \text{and} \quad u_y(x,\pi,t) \overset{⑤}{=} 0 \quad \text{for } 0 \le x \le \pi \text{ and } t \ge 0.$$

We seek nontrivial solutions of ①-②-③-④-⑤ of the form $u(x,y,t) = X(x)Y(y)T(t)$.

① $\Rightarrow$ $X(x)Y(y)T'(t) - X''(x)Y(y)T(t) - X(x)Y''(y)T(t) = 0$ so

$$\frac{T'(t)}{T(t)} - \frac{X''(x)}{X(x)} - \frac{Y''(y)}{Y(y)} = 0 \quad \Rightarrow \quad \frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} - \frac{Y''(y)}{Y(y)} = -\lambda.$$

Also $\dfrac{Y''(y)}{Y(y)} = \dfrac{T'(t)}{T(t)} + \lambda = -\mu.$

②-③ $\Rightarrow$ $X'(0)Y(y)T(t) = 0 = X'(\pi)Y(y)T(t)$,

④-⑤ $\Rightarrow$ $Y'(0)X(x)T(t) = 0 = Y'(\pi)X(x)T(t)$.

4 $\therefore \begin{cases} X'' + \lambda X = 0, & X'(0) = 0 = X' \\ Y'' + \mu Y = 0, & Y'(0) = 0 = Y' \\ T' + (\mu + \lambda)T = 0 \end{cases}$

4

2

3 Eigenvalues: $\lambda_\ell = \ell^2$ and $\mu_m = m^2$ $(\ell, m = 0, 1, 2, \ldots)$

Also, $T_{\ell,m}(t) = e^{-(\ell^2+m^2)t}$. 2

3 Eigenfunctions: $X_\ell(x) = \cos(\ell x)$ and $Y_m(y) = \cos(my)$

3 Therefore, a formal solution of ①-②-③-④-⑤ is $u(x,y,t) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} A_{\ell,m}\cos(\ell x)\cos(my)e^{-(\ell^2+m^2)t}$

We need to determine the coefficients so ⑥ is satisfied:

$$\cos^2(x)\cos^2(y) = u(x,y,0) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} A_{\ell,m}\cos(\ell x)\cos(my) \quad \text{for all } 0 \le x \le \pi,\ 0 \le y$$

Using the identity $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$ twice on the LHS and expanding gives

3

$$\frac{1}{4} + \frac{1}{4}\cos(2x) + \frac{1}{4}\cos(2y) + \frac{1}{4}\cos(2x)\cos(2y) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} A_{\ell,m}\cos(\ell x)\cos(my).$$

3 By inspection, we may take $A_{0,0} = \frac{1}{4}$, $A_{2,0} = \frac{1}{4}$, $A_{0,2} = \frac{1}{4}$, $A_{2,2} = \frac{1}{4}$, and all other $A_{\ell,m} = 0$.

3 $\therefore$ $\boxed{u(x,y,t) = \frac{1}{4} + \frac{1}{4}\cos(2x)e^{-4t} + \frac{1}{4}\cos(2y)e^{-4t} + \frac{1}{4}\cos(2x)\cos(2y)e^{-8t}}$.

6.(33 pts.) Solve the Poisson equation $\nabla^2 u = 1$ in the spherical shell $1 < r < 3$, given that $u = 0$ on $r = 1$ and $\frac{\partial u}{\partial r} = 0$ on $r = 3$.

Note: You may find useful the fact that the three-dimensional Laplacian in spherical polar coordinates is

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin(\phi)}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial u}{\partial \phi}\right) + \frac{1}{r^2 \sin^2(\phi)}\frac{\partial^2 u}{\partial \theta^2}.$$

By rotational invariance of the p.d.e., boundary conditions, and the domain on which we solve the problem, we may assume that the solution $u$ is a function of $r$ only and is independent

5 of $\varphi$ and $\theta$. Therefore the p.d.e. reduces to $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = 1 \Rightarrow \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = r^2$

5

6 + 1 + 6 $\Rightarrow r^2 \frac{\partial u}{\partial r} = \frac{r^3}{3} + c_1 \Rightarrow \frac{\partial u}{\partial r} = \frac{r}{3} + \frac{c_1}{r^2} \Rightarrow u = \frac{r^2}{6} - \frac{c_1}{r} + c_2$

5 $0 = \left.\frac{\partial u}{\partial r}\right|_{r=3} = \frac{3}{3} + \frac{c_1}{9} \Rightarrow c_1 = -9.$ $\quad 0 = \left.u\right|_{r=1} = \frac{1}{6} - \overset{-9}{\cancel{c_1}} + c_2 \Rightarrow c_2 = \frac{-55}{6}$

5 $\therefore \boxed{u(r, \varphi, \theta) = \frac{r^2}{6} + \frac{9}{r} - \frac{55}{6} = \frac{r^2 - 1}{6} + 9\left(\frac{1}{r} - 1\right)}.$

17 pts.

Bonus(33 pts.): (a) Use Fourier transform methods to show that, under appropriate hypotheses on the function $f$, a solution to the inhomogeneous wave equation

$$u_{tt} - c^2 u_{xx} \overset{(1)}{=} f(x,t) \quad \text{for} \quad -\infty < x < \infty \quad \text{and} \quad -\infty < t < \infty$$

satisfying

$$u(x,0) \overset{(2)}{=} 0 \overset{(3)}{=} u_t(x,0) \quad \text{for} \quad -\infty < x < \infty$$

is given by

$$u(x,t) = \frac{1}{2c} \int_0^t \left( \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(y,\tau)\,dy \right) d\tau.$$

16 pts. (b) Use the formula in part (a) to find a solution to

$$u_{tt} - u_{xx} = xt \quad \text{for} \quad -\infty < x < \infty \quad \text{and} \quad -\infty < t < \infty$$

which satisfies $u(x,0) = 0 = u_t(x,0)$ for $-\infty < x < \infty$.

2 (a) $\mathcal{F}(u_{tt} - c^2 u_{xx})(\xi) = \mathcal{F}(f(\cdot,t))(\xi) \Rightarrow \frac{\partial^2}{\partial t^2}\mathcal{F}(u)(\xi) + c^2\xi^2\mathcal{F}(u)(\xi) = F(\xi,t)$

Recall that the solution of $y'' + \lambda^2 y = r(t)$ is $y = y_h + y_p$ where $y_h = c_1\cos(\lambda t) + c_2\sin(\lambda t)$

and $y_p = v_1(t)\cos(\lambda t) + v_2(t)\sin(\lambda t)$ $\quad v_1(t) = \int_0^t \frac{-\sin(\lambda\tau)r(\tau)}{\lambda}\,d\tau$, $v_2(t) = \int_0^t \frac{\cos(\lambda\tau)r(\tau)}{\lambda}\,d\tau$

Thus $y_p = \frac{1}{\lambda}\int_0^t \left[-\sin(\lambda\tau)\cos(\lambda t) + \cos(\lambda\tau)\sin(\lambda t)\right] r(\tau)\,d\tau = \frac{1}{\lambda}\int_0^t \sin(\lambda(t-\tau))r(\tau)\,d\tau.$

Applying this to the D.E. above yields

6 $$\mathcal{F}(u)(\xi) = c_1(\xi)\cos(c\xi t) + c_2(\xi)\sin(c\xi t) + \int_0^t \frac{\sin(c\xi(t-\tau))F(\xi,\tau)}{c\xi}\,d\tau.$$

$$(2) \Rightarrow \quad 0 = \mathcal{F}(u)(\xi)\Big|_{t=0} = c_1(\xi).$$

$$(3) \Rightarrow \quad 0 = \mathcal{F}(u_t)(\xi)\Big|_{t=0} = c_2(\xi)c\xi \quad \text{so} \quad c_2(\xi) = 0.$$

2 $\therefore \mathcal{F}(u)(\xi) = \frac{1}{c}\int_0^t \frac{\sin(c\xi(t-\tau))}{\xi}F(\xi,\tau)\,d\tau$. By entry (A) in the table of

1 Fourier transforms (with $b = c(t-\tau)$) yields $\sqrt{\frac{\pi}{2}}\,\mathcal{F}\chi_{(-c(t-\tau),\,c(t-\tau))}(\xi) = \frac{\sin(c\xi(t-\tau))}{\xi}$

1 $$\therefore \mathcal{F}(u)(\xi) = \frac{1}{c}\int_0^t \mathcal{F}\left(\sqrt{\tfrac{\pi}{2}}\,\chi_{(-c(t-\tau),\,c(t-\tau))}\right)(\xi)\,\mathcal{F}\left(f(\cdot,\tau)\right)(\xi)\,d\tau$$

2 $$= \frac{1}{c}\int_0^t \frac{1}{\sqrt{2\pi}}\mathcal{F}\left(\sqrt{\tfrac{\pi}{2}}\,\chi_{(-c(t-\tau),\,c(t-\tau))} * f(\cdot,\tau)\right)(\xi)\,d\tau$$

Final Exam

Summer 2004

$\mu = 142.4 \quad 150.1$

$\sigma = 40.8 \quad 29.0$

} Without the 35 score.

Distribution of Scores

| | |
|---|---|
| 194 | A |
| 193 | A |
| 190 | A |
| 178 | A |
| 159 | B |
| 152 | B |
| 148 | B |
| ~~145~~ | |
| 142 | B |
| 139 | B |
| 137 | B |
| 129 | B |
| 123 | B |
| 123 | B |
| 101 | C |
| 35 | D |

Distribution of Grades

| | |
|---|---|
| A | 4 |
| B | 9 |
| C | 1 |
| D | 1 |
| F | 0 |