Exercises for Fourier Transform Methods for Solving PDE's

1. (a) Use Fourier transform methods to derive d'Alembert's solution to the initial value problem for the 1-D wave equation

\[ u_{tt} - c^2 u_{xx} = 0 \quad \text{for} \quad -\infty < x < \infty, \quad -\infty < t < \infty, \]
\[ u(x,0) = \phi(x) \quad \text{and} \quad u_t(x,0) = \psi(x) \quad \text{for} \quad -\infty < x < \infty. \]

(b) What assumptions on \( \phi \) and \( \psi \) do you make in order for the derivation in part (a) to be rigorous?

2. Solve problem # 16 in Sec. 2.4 by Fourier transform methods.

3. Solve problem # 17 in Sec. 2.4 by Fourier transform methods.

4. Solve problem # 18 in Sec. 2.4 by Fourier transform methods.

5. Let \( f \) be a piecewise-continuous absolutely integrable function on \( -\infty < x < \infty \).

(a) Use Fourier transform methods to solve the 2-D Laplace equation

\[ u_{xx} + u_{yy} = 0 \quad \text{in the upper halfplane} \quad -\infty < x < \infty, \quad 0 < y < \infty \]

subject to the boundary condition
\[ u(x,0) = f(x) \quad \text{for} \quad -\infty < x < \infty \]

and the decay condition
\[ u(x,y) \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty. \]

(b) Let \( f(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{otherwise}. \end{cases} \) Compute an explicit formula for the solution \( u = u(x,y) \) in part (a).
Exercises for Fourier Transform Methods

1. (a) Use Fourier transform methods to derive d'Alembert's solution to the initial value problem for the one-dimensional wave equation

\[ u_{tt} - c^2 u_{xx} = 0 \quad \text{for} \quad -\infty < x < \infty, \quad -\infty < t < \infty, \]

\[ u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x) \quad \text{for} \quad -\infty < x < \infty. \]

(b) What assumptions on \( f \) and \( g \) do you make in order for the derivation in part (a) to be rigorous?

He will make use of the following result in part (a).

**FACT:** Let \( f \) be a piecewise-continuous absolutely integrable function on \((-\infty, \infty)\) such that \( \hat{f}(0) = 0 \), and let

\[ F(x) = \int_{-\infty}^{x} f(y) \, dy, \quad x \in (-\infty, \infty). \]

Then \( \hat{F}(\xi) = \frac{\hat{f}(\xi)}{i\xi} \) for \( \xi \neq 0 \).

**Proof of FACT:** If \( \xi \to 0 \) then

\[ \hat{F}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) e^{-ix\xi} \, dx \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{x} f(y) \, dy \right) e^{-ix\xi} \, dx \]

\[ = \frac{1}{\sqrt{2\pi}} \left[ \left( \int_{-\infty}^{\infty} f(y) \, dy \right) e^{-i\xi x} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-i\xi x} \hat{f}(x) \, dx \]

But \( \lim_{x \to -\infty} \int_{x}^{\infty} f(y) \, dy = 0 \), \( \lim_{x \to \infty} \int_{-\infty}^{x} f(y) \, dy = \int_{-\infty}^{\infty} f(y) \, dy = 0 \),

and \( \left| \frac{e^{-i\xi x}}{-i\xi} \right| = \frac{1}{\xi} \) for all real \( x \). It follows that...
1. (cont.) \[ \hat{F}(\xi) = \frac{1}{i\xi} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \, dx = \frac{\hat{f}(\xi)}{i\xi} \].

(a) \[ \hat{f}(u_{tt} - c^2 u_{xx}) = \hat{f}(0) \]

\[ \frac{\partial^2}{\partial t^2} \hat{f}(u) + c^2 \frac{\partial^2}{\partial x^2} \hat{f}(u) = 0 \]

\[ \hat{f}(u) = c_1(\xi) \cos(c \xi t) + c_2(\xi) \sin(c \xi t) \]

\[ \hat{\phi}(\xi) = \hat{f}(u(\cdot, 0)) = c_1(\xi) \]

\[ \hat{\psi}(\xi) = \left. \frac{\partial}{\partial t} \hat{f}(u) \right|_{t=0} = -c_3 c_1(\xi) \sin(c \xi t) + c_3 c_2(\xi) \cos(c \xi t) \]

\[ \hat{\psi}(\xi) = c_3 c_2(\xi) \]

\[ \therefore \hat{f}(u) = \hat{\phi}(\xi) \cos(c \xi t) + \frac{\hat{\psi}(\xi)}{i c_3 \xi} \sin(c \xi t) \]

\[ = \frac{1}{2} \hat{\phi}(\xi) e^{ic \xi t} + \frac{1}{2} \hat{\phi}(\xi) e^{-ic \xi t} + \frac{\hat{\psi}(\xi)}{i c_3 \xi} \left( \frac{e^{ic \xi t} - e^{-ic \xi t}}{2i} \right) \]

\[ = \frac{1}{2} \hat{\phi}(\xi) e^{ic \xi t} + \frac{1}{2} \hat{\phi}(\xi) e^{-ic \xi t} + \frac{1}{2c} \hat{\psi}(\xi) e^{ic \xi t} - \frac{1}{2c} \hat{\psi}(\xi) e^{-ic \xi t} \]

If \[ \overline{F}(x) = \int_{-\infty}^{\infty} F(y) dy \] then \( \hat{\overline{F}}(\xi) = \frac{\hat{F}(\xi)}{i \xi} \) by FACT. Thus

\[ \hat{F}(u) = \frac{1}{2} \hat{\phi}(\xi) e^{ic \xi t} + \frac{1}{2} \hat{\phi}(\xi) e^{-ic \xi t} + \frac{1}{2c} \hat{\psi}(\xi) e^{ic \xi t} - \frac{1}{2c} \hat{\psi}(\xi) e^{-ic \xi t} \]

Fix the time \( t \) by the "shifting on the x-axis" result (#4 on the Exercises for Fourier Transforms),

\[ f_1(x) = \phi(x + ct) \] has Fourier transform \( \hat{f}_1(\xi) = e^{ic \xi t} \hat{\phi}(\xi) \);

\[ f_2(x) = \phi(x - ct) \] " " " " \( \hat{f}_2(\xi) = e^{-ic \xi t} \hat{\phi}(\xi) \);
Exercises for Fourier Transform Methods (cont.)

\[ g_1(x) = \mathcal{F}(x+ct) \] has Fourier transform \[ \hat{g}_1(\xi) = e^{isct} \mathcal{F}(\xi); \]
\[ g_2(x) = \mathcal{F}(x-ct) \] \[ \hat{g}_2(\xi) = e^{-isct} \mathcal{F}(\xi). \]

Substituting these relations into (*) gives

\[ \mathcal{F}(u) = \mathcal{F}
\left( \frac{1}{2} f_1 + \frac{1}{2} f_2 + \frac{1}{2c} g_1 - \frac{1}{2c} g_2 \right), \]

and the uniqueness theorem implies (for fixed \( t \) and any real \( x \))

\[ u(x,t) = \frac{1}{2} f_1(x) + \frac{1}{2} f_2(x) + \frac{1}{2c} g_1(x) - \frac{1}{2c} g_2(x) \]

\[ = \frac{1}{2} \mathcal{F}(x+ct) + \frac{1}{2} \mathcal{F}(x-ct) + \frac{1}{2c} \mathcal{F}(x+ct) - \frac{1}{2c} \mathcal{F}(x-ct) \]

\[ = \frac{1}{2} \left[ \mathcal{F}(x+ct) + \mathcal{F}(x-ct) \right] + \frac{1}{2c} \left[ \int_{-\infty}^{x+ct} \mathcal{F}(y) dy - \int_{-\infty}^{x-ct} \mathcal{F}(y) dy \right]. \]

\[ u(x,t) = \frac{1}{2} \left[ \mathcal{F}(x+ct) + \mathcal{F}(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \mathcal{F}(y) dy \]

(b) In order for the function \( u = u(x,t) \) above to satisfy the P.D.E., it is clear that we must have \( \mathcal{F} \in C^2(-\infty, \infty) \) and \( f \in C^1(-\infty, \infty) \). In the derivation by Fourier transform methods in part (a), we applied FACT with \( f = \mathcal{F} \). Therefore, \( \mathcal{F} \) must be absolutely integrable on \( (-\infty, \infty) \). Since we take the Fourier transform of \( \mathcal{F} \), it is natural to require that \( \mathcal{F} \) be absolutely integrable on \( (-\infty, \infty) \). Finally, we interchange integration and differentiation when we write \( \frac{\partial^2}{\partial t^2} \mathcal{F}(u) = \mathcal{F}(u_{tt}) \). Thus, by
1(b) (cont.) Theorem 2 of A.3 (see p. 390), it is natural to require that $\phi''$ and $\psi'$ be absolutely integrable on $(-\infty, \infty)$.

2. Solve problem 416 in Sec. 2.4 by Fourier Transform methods.

"Solve the diffusion equation with constant dissipation:

$$u_t - ku_{xx} + bu = 0 \quad \text{for } -\infty < x < \infty, \ 0 < t < \infty,$$

with $u(x, 0) = \phi(x)$ for $-\infty < x < \infty$. Here $b > 0$ is constant."

Taking the Fourier transform of both sides of the PDE with respect to the variable $x$ yields

$$\mathcal{F}(u_t - ku_{xx} + bu) = \mathcal{F}(0)$$

$$\frac{2}{\pi} \mathcal{F}(u) - k(-s^2)\mathcal{F}(u) + b\mathcal{F}(u) = 0.$$  

$$\therefore \mathcal{F}(u) = c(s)e^{-(-k s^2 + b)t}.$$  

$$\mathcal{F}(\phi) = \mathcal{F}(u(\cdot, 0)) = c(s)e^0 = c(s),$$

$$\therefore \mathcal{F}(u) = \mathcal{F}(\phi)e^{-(-k s^2 + b)t}.$$  

Using formula I in the Table of Fourier Transforms with $kt = \frac{1}{4a}$ yields

$$\frac{1}{\sqrt{2\pi kt}} \mathcal{F}(e^{-\frac{(s)^2}{4kt}}) e^{-bt} = \frac{1}{\sqrt{2\pi kt}} e^{-\frac{-k t s^2 - bt}{\sqrt{2kt}}} e^{-(k s^2 + b)t}$$

Substituting this expression into (*) produces...
2. (cont.) \( \mathcal{F}(u)_t = \mathcal{F}(q) \frac{1}{\sqrt{2\pi kt}} \mathcal{F} \left( e^{-\frac{(\cdot)^2}{4kt}} \right) e^{-bt} \)

\[
= e^{-bt} \frac{1}{\sqrt{2\pi}} \mathcal{F} \left( q \star e^{-\frac{(\cdot)^2}{4kt}} \right)
\]

\[
= \mathcal{F} \left( \frac{e^{-bt}}{\sqrt{4kt}} \right) q \star e^{-\frac{(\cdot)^2}{4kt}}
\]

By the uniqueness theorem (for fixed \( t > 0 \) and any real \( x \)) it follows that

\[
u(x, t) = \frac{e^{-bt}}{\sqrt{4\pi kt}} (q \star e^{-\frac{(\cdot)^2}{4kt}})(x),
\]

i.e.

\[
u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} q(y) \, dy
\]

3. Solve problem #17 in Sec. 2.4 by Fourier transform methods.

"Solve the diffusion equation with variable dissipation:

\[
u_t - ku_{xx} + bt \nu = 0
\]

for \(-\infty < x < \infty, \ 0 < t < \infty, \) with \( u(x, 0) = q(x) \) for \(-\infty < x < \infty; \)

here \( b > 0 \) is a constant."

Taking the Fourier transform (with respect to \( x \)) of both sides of the PDE yields

\[
\mathcal{F}(u_t) - k \mathcal{F}(u_{xx}) + bt^2 \mathcal{F}(u) = 0
\]

\[
\frac{2}{\partial t} \mathcal{F}(u) - k(i\xi)^2 \mathcal{F}(u) + bt^2 \mathcal{F}(u) = 0
\]
Exercises for Fourier Transform Methods (cont.)

3. (cont.) \( \frac{\partial^2 F(u)}{\partial t^2} + (k^2 u^2 + bt^2) F(u) = 0 \).

Separating variables and integrating produces

\[ \ln F(u) = -k^2 t - \frac{bt^3}{3} + c(s) \]

or

\[ F(u) = A(s) e^{\frac{-k^2 t^2}{3} - \frac{bt^3}{3}}. \]  

(where \( A(s) = e^{c(s)} \)).

Applying the initial condition we have

\[ F(\phi) = F(u(\cdot, 0)) = A(s) e^0 = A(s) \]

so

\[ F(u) = F(\phi) e^{\frac{-k^2 t^2}{3} - \frac{bt^3}{3}}. \]  

Applying formula 1: \( F(e^{-a(\cdot)^2}) = \frac{e^{-\frac{a^2}{4a}}}{\sqrt{2a}} \), with \( kT = \frac{1}{4a} \) (that is, \( a = \frac{1}{4kt} \)) gives \( F(e^{-\frac{c^2}{4kt}}) = \sqrt{2kt} e^{\frac{-k^2 t^2}{3}} \). Substituting this expression into (†), we find

\[ F(u) = F(\phi) F(e^{-\frac{c^2}{4kt}}) \cdot \frac{e^{-\frac{bt^3}{3}}}{\sqrt{2kt}}. \]

Using the convolution formula \( F(f * g) = \sqrt{2\pi} F(f) F(g) \) we have

\[ F(u) = \frac{1}{\sqrt{2\pi}} F(\phi * e^{-\frac{c^2}{4kt}}) \cdot e^{\frac{-k^2 t^2}{3}} \]

\[ = F(\frac{e^{-\frac{kT^2}{3}}}{\sqrt{4\pi KT}} \phi * e^{-\frac{c^2}{4kt}}). \]

By the uniqueness theorem (for fixed \( t > 0 \) and any real \( x \))

\[ u(x, t) = \frac{e^{-bt^3/3}}{\sqrt{4\pi KT}} \phi * e^{-\frac{c^2}{4kt}}(x) \]

i.e.

\[ u(x, t) = \frac{e^{-bt^3/3}}{\sqrt{4\pi KT}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y) dy. \]
Exercises for Fourier Transform Methods (cont.)

4. Solve problem #18 in Sec. 2.4 by Fourier transform methods.

"Solve the heat equation with convection:

\[ u_t - ku_{xx} + V u_x = 0 \quad \text{for} \ -\infty < x < \infty, \ 0 < t < \infty, \]

with \( u(x,0) = f(x) \) for \(-\infty < x < \infty\), where \( V \) is a constant."

Taking the Fourier transform (with respect to \( x \)) of the PDE yields

\[
\mathcal{F}(u_t) - k \mathcal{F}(u_{xx}) + V \mathcal{F}(u_x) = 0
\]

\[
\frac{\partial \mathcal{F}(u)}{\partial t} - k(i\xi)^2 \mathcal{F}(u) + V(i\xi) \mathcal{F}(u) = 0
\]

\((\dagger)\)

\[
\frac{\partial}{\partial t} \mathcal{F}(u) + (k\xi^2 + iV\xi) \mathcal{F}(u) = 0.
\]

An integrating factor for this linear first-order equation in \( t \) (for fixed \( \xi \)) is

\[
\int (k\xi^2 + iV\xi) dt = e^{(k\xi^2 + iV\xi)t}.
\]

Multiplying \((\dagger)\) by the integrating factor and using the product rule for derivatives gives

\[
\frac{\partial}{\partial t} \left\{ \mathcal{F}(u) e^{(k\xi^2 + iV\xi)t} \right\} = 0,
\]

and algebra

whereupon integration yields

\[
\mathcal{F}(u) = c(\xi) e^{(k\xi^2 + iV\xi)t}.
\]

Applying the initial condition, we have

\[
\mathcal{F}(f) = \mathcal{F}(u(\cdot,0)) = c(\xi) e^{0} = c(\xi).
\]

Thus \((\dagger)\) \( \mathcal{F}(u) = \mathcal{F}(f) \cdot e^{-\frac{k\xi^2}{2} - iVt\xi} \).
Exercises for Fourier Transform Methods (cont.)

4. (cont.) As in problems 2 and 3, \( \mathcal{F}(e^{-\frac{(\cdot)^2}{4kt}}) = \sqrt{2k\pi} e^{-\frac{ktx^2}{4}} \), so substituting in (*) we have

\[
\mathcal{F}(u) = \mathcal{F}(\varphi) \mathcal{F}\left( \frac{1}{\sqrt{2k\pi}} e^{-\frac{(\cdot)^2}{4kt}} \right) e^{-i\sqrt{Vt} \cdot \varphi},
\]

and using the convolution formula as in problems 2 and 3,

\[
(\ast\ast) \quad \mathcal{F}(u) = e^{-i\sqrt{Vt} \cdot \varphi} \mathcal{F}\left( \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(\cdot)^2}{4kt}} \ast \varphi \right).
\]

Applying the shifting formula on the x-axis (\#4 on Exercises for Fourier Transforms): \( \mathcal{F}(f(\cdot-a)) = e^{-i\sqrt{\pi}a} \mathcal{F}(\varphi) \), with \( a = Vt \), (\ast\ast) becomes

\[
\mathcal{F}(u) = \mathcal{F}\left( \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(\cdot-Vt)^2}{4kt}} \ast \varphi \right).
\]

The uniqueness theorem then implies (for fixed \( t > 0 \) and any real \( x \))

\[
u(x,t) = \frac{1}{\sqrt{4\pi kt}} (e^{-\frac{(x-Vt)^2}{4kt}} \ast \varphi)(x)
\]

i.e.

\[
u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y-Vt)^2}{4kt}} \varphi(y) \, dy.
\]

5. Let \( f \) be an absolutely integrable, piecewise-continuous function on \(-\infty < x < \infty\).

(a) Use Fourier transform methods to solve the 2-D Laplace equation \( u_{xx} + u_{yy} = 0 \) in the upper halfplane \(-\infty < x < \infty, 0 < y < \infty\), subject to the boundary condition \( u(x,0) = f(x) \) for \(-\infty < x < \infty\) and
Exercises for Fourier Transform Methods (cont.)

5. (cont.) the decay condition \( u(x, y) \to 0 \) as \( x^2 + y^2 \to \infty \).

(b) Let
\[
\begin{align*}
f(x) &= \begin{cases} 
1 & \text{if } |x| < 1, \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]

Compute an explicit formula for the solution \( u = u(x, y) \) in part (a).

(a) We take the Fourier transform (with respect to \( x \)) of the PDE:
\[
\mathcal{F}(u_{xx}) + \mathcal{F}(u_{yy}) = 0
\]
\[
-\xi^2 \mathcal{F}(u) + \frac{\partial^2}{\partial y^2} \mathcal{F}(u) = 0.
\]

(4)
\[
\mathcal{F}(u) = c_1(\xi) e^{\xi y} + c_2(\xi) e^{-\xi y}.
\]

Applying the boundary condition yields

(5)
\[
\mathcal{F}(f)(\xi) = \mathcal{F}(u(., 0))(\xi) = c_1(\xi) e^0 + c_2(\xi) e^0 = c_1(\xi) + c_2(\xi)
\]

for \(-\infty < \xi < \infty\). The decay condition implies

(6)
\[
0 = \lim_{|y| \to \infty} \mathcal{F}(u) = \lim_{|y| \to \infty} \left( c_1(\xi) e^{\xi y} + c_2(\xi) e^{-\xi y} \right)
\]

for \(-\infty < \xi < \infty\).

Suppose \( \xi > 0 \); let \( y \to +\infty \) in (6) and observe that we must have \( c_1(\xi) = 0 \) if (6) is to be satisfied. Suppose \( \xi < 0 \); let \( y \to +\infty \) in (6) and observe that this time we must have \( c_2(\xi) = 0 \). Using these relations in conjunction with (5) yields

(7)
\[
c_1(\xi) = \begin{cases} 
\mathcal{F}(f)(\xi) & \text{if } \xi < 0, \\
0 & \text{if } \xi > 0,
\end{cases}
\]
\[
c_2(\xi) = \begin{cases} 
0 & \text{if } \xi < 0, \\
\mathcal{F}(f)(\xi) & \text{if } \xi > 0.
\end{cases}
\]
Exercises for Fourier Transform Methods (cont.)

5. (cont.) By (t) and (ttt),

\[ F(u) = \begin{cases} 
\mathcal{F}(f)(\xi) e^{i\xi y} & \text{if } \xi < 0, \\
\mathcal{F}(f)(\xi) e^{-i\xi y} & \text{if } \xi > 0,
\end{cases} \]

or equivalently,

\[ (\text{tttt}) \quad F(u) = \mathcal{F}(f)(\xi) e^{-|\xi|y}. \]

Applying formula C for Fourier transforms:

\[ \mathcal{F}\left( \frac{1}{(\cdot)^2 + a^2} \right) = \sqrt{\frac{\pi}{2}} \frac{e^{-a|\xi|}}{a} \]

with \( a = y > 0 \), we find that \( \mathcal{F}\left( \frac{\sqrt{\frac{2}{\pi}} \cdot y}{(\cdot)^2 + y^2} \right) = e^{-|\xi|y} \).

Substituting in (tttt) and using the convolution formula

\[ \mathcal{F}(f \ast g) = \sqrt{2\pi} \mathcal{F}(f) \mathcal{F}(g), \]

we have

\[ F(u) = \mathcal{F}(f) \mathcal{F}\left( \frac{\sqrt{\frac{2}{\pi}} \cdot y}{(\cdot)^2 + y^2} \right) = \sqrt{\frac{\pi}{2}} \mathcal{F}\left( f \ast \frac{\sqrt{\frac{2}{\pi}} \cdot y}{(\cdot)^2 + y^2} \right), \]

whereupon the uniqueness theorem (for fixed \( y > 0 \) and all real \( x \)) gives

\[ u(x,y) = \frac{1}{\pi} \left( f \ast \frac{y}{(\cdot)^2 + y^2} \right)(x) \]

i.e.

\[ u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y f(s) \, ds}{(x-s)^2 + y^2}. \]

(b) If \( f(x) = \begin{cases} 
1 & \text{if } |x| < 1, \\
0 & \text{otherwise},
\end{cases} \)

then the solution to the problem in part (a) is
5. (cont.)  
\[ u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y f(s) ds}{(x-s)^2 + y^2} = \frac{1}{\pi} \int_{-1}^{1} \frac{y \cdot 1 ds}{(x-s)^2 + y^2} \]

\[ = \frac{1}{\pi} \int_{-1}^{1} \frac{\frac{y}{y} ds}{(x-s)^2 + 1} \cdot \text{Let } p = \frac{s-x}{y}. \text{ Then } dp = \frac{1}{y} ds \text{ so} \]

\[ u(x, y) = \frac{1}{\pi} \int_{-1}^{1} \frac{dp}{p^2 + 1} = \frac{1}{\pi} \left[ \text{Arctan} \left( \frac{1-x}{y} \right) - \text{Arctan} \left( \frac{-1-x}{y} \right) \right]. \]

\[ \therefore \quad u(x, y) = \frac{1}{\pi} \left[ \text{Arctan} \left( \frac{1-x}{y} \right) + \text{Arctan} \left( \frac{1+x}{y} \right) \right]. \]