Exercises for Fourier Transforms

1. Derive formula B in the table of Fourier transforms.

2. Obtain formula A from formula B in the table of Fourier transforms.

3. Find the Fourier transform of $f(x) = e^{-|x|}$.

4 (Shifting on the x-axis) Show that if f has a Fourier transform $\hat{f}$, then so does the translate of f by a given by $f_a(x) = f(x-a)$, and

$$\hat{f}_a(\xi) = e^{-i\xi a} \cdot \hat{f}(\xi).$$

5. Using the result of problem ~~3~~ 4 and formula D in the table of Fourier transforms, obtain the Fourier transform of the function

$$g(x) = \text{maximum of } \{0,\ b - |x|\} = \begin{cases} x + b & \text{if } -b < x \le 0, \\ b - x & \text{if } 0 < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

6. (Shifting on the ξ-axis) Show that if f has a Fourier transform $\hat{f}$, then $\hat{f}(\xi - a)$ is the Fourier transform of $e^{iax} f(x)$.

7. Using the result of problem 6, obtain formula G in the table of Fourier transforms from formula A.

8. Using the result of problem 6, obtain formula H in the table of Fourier transforms from formula B.

9. Verify formula C in the table of Fourier transforms. (Hint: ~~Use~~ Generalize the result of problem 3 and ~~then~~ use the inversion formula.)

10. Verify formula J in the table of Fourier transforms. (Hint: Use formula A in the table of Fourier transforms and the inversion formula.)

11. Verify formula D in the table of Fourier transforms. (Hint: Use convolution and formula B in the table of Fourier transforms with $c = 0$ and $d = b$.)

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$
A.	$\begin{cases} 1 & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin(b\xi)}{\xi}$
B.	$\begin{cases} 1 & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{e^{-ic\xi} - e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\dfrac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\dfrac{\pi}{2}}\,\dfrac{e^{-a\lvert\xi\rvert}}{a}$
D.	$\begin{cases} x & \text{if } 0 < x \leq b, \\ 2b - x & \text{if } b < x < 2b, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{-1 + 2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases} e^{-ax} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (a > 0)$	$\dfrac{1}{(a + i\xi)\sqrt{2\pi}}$
F.	$\begin{cases} e^{ax} & \text{if } b < x < c, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{e^{(a-i\xi)c} - e^{(a-i\xi)b}}{(a - i\xi)\sqrt{2\pi}}$
G.	$\begin{cases} e^{iax} & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin(b(\xi-a))}{\xi-a}$
H.	$\begin{cases} e^{iax} & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{i}{\sqrt{2\pi}}\,\dfrac{e^{ic(a-\xi)} - e^{id(a-\xi)}}{a - \xi}$
I.	$e^{-ax^2} \quad (a > 0)$	$\dfrac{1}{\sqrt{2a}}\,e^{-\xi^2/(4a)}$
J.	$\dfrac{\sin(ax)}{x} \quad (a > 0)$	$\begin{cases} \sqrt{\pi/2} & \text{if } \lvert\xi\rvert < a, \\ 0 & \text{if } \lvert\xi\rvert > a. \end{cases}$

Exercises for Fourier Transforms.

1. Derive formula B in the table of Fourier transforms.

Consider $f(x) = \begin{cases} 1 & \text{if } c<x<d, \\ 0 & \text{otherwise.} \end{cases}$

Then $$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx = \frac{1}{\sqrt{2\pi}} \int_c^d 1 \cdot e^{-i\xi x}\, dx = \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-i\xi x}}{-i\xi} \right) \Bigg|_{x=c}^{x=d}$$

$$\boxed{\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-i\xi c} - e^{-i\xi d}}{i\xi}}$$

This formula is valid if $\xi \neq 0$. If $\xi = 0$ then $\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_c^d 1 \cdot dx = \frac{d-c}{\sqrt{2\pi}}$.

Note that $$\lim_{\xi \to 0} \hat{f}(\xi) = \lim_{\xi \to 0} \frac{e^{-i\xi c} - e^{-i\xi d}}{i\xi\sqrt{2\pi}} = \frac{-ic + id}{i\sqrt{2\pi}} = \hat{f}(0).$$

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2. Obtain formula A from formula B in the table of Fourier transforms.

Consider $f(x) = \begin{cases} 1 & \text{if } -b<x<b, \\ 0 & \text{otherwise.} \end{cases}$

Taking $c=-b$ and $d=b$ in the formula B derived in problem 1, we have that

$$\hat{f}(\xi) = \frac{2}{\sqrt{2\pi}} \cdot \frac{e^{i\xi b} - e^{-i\xi b}}{2i\xi} = \boxed{\sqrt{\frac{2}{\pi}} \cdot \frac{\sin(b\xi)}{\xi}}.$$

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3. Find the Fourier transform of $f(x) = e^{-|x|}$.

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-i\xi x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{x} \cdot e^{-i\xi x}\, dx + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x} \cdot e^{-i\xi x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(1-i\xi)x}\, dx + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{(-1-i\xi)x}\, dx$$

3. (cont.) $$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \cdot \left(\frac{e^{(1-i\xi)x}}{1-i\xi} \right) \Bigg|_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \cdot \left(\frac{e^{(-1-i\xi)x}}{-1-i\xi} \right) \Bigg|_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{1-i\xi} + \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{1+i\xi}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1+i\xi + 1-i\xi}{(1-i\xi)(1+i\xi)} \right]$$

$$= \boxed{\sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+\xi^2}}$$

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4. (Shifting on the x-axis) Show that if f has a Fourier transform $\hat{f}$, then so does the translate of f by a given by $f_a(x) = f(x-a)$, and $\hat{f}_a(\xi) = e^{-i\xi a}\hat{f}(\xi)$.

By definition, $$\hat{f}_a(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_a(x) e^{-i\xi x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{-i\xi x}\, dx .$$

Making the change of variables $y = x-a$ in this integral, we find

$$\hat{f}_a(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-i\xi(y+a)}\, dy$$

$$= e^{-i\xi a} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-i\xi y}\, dy$$

$$= \boxed{e^{-i\xi a}\hat{f}(\xi)}$$

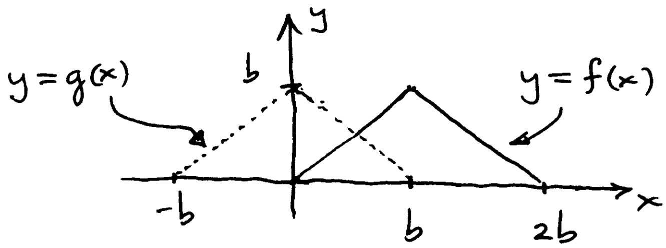
Exercises for Fourier Transforms (cont.)

5. Using the result of problem 4 and formula D in the table of Fourier transforms, obtain the Fourier transform of the function

$$g(x) = \max\{0,\, b-|x|\} = \begin{cases} b+x & \text{if } -b < x \le 0, \\ b-x & \text{if } \ 0 < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the graphs of g (above) and the function f in formula D:

$$f(x) = \begin{cases} x & \text{if } 0 < x \le b, \\ 2b-x & \text{if } b < x < 2b, \\ 0 & \text{otherwise.} \end{cases}$$



It is apparent that the graph of g is the shift of the graph of f to the left by b units, and hence $g(x) = f(x+b)$ for all x. That is, $g(x) = f_{-b}(x)$ in the notation of problem 4.

Using the shifting formula of problem 4 and formula D in the table of Fourier transforms, it follows that

$$\hat{g}(\xi) = \widehat{f_{-b}}(\xi) = e^{i\xi b}\hat{f}(\xi) = e^{i\xi b}\cdot\left[\frac{-1+2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}\right]$$

$$\hat{g}(\xi) = \frac{-e^{i\xi b} + 2 - e^{-ib\xi}}{\xi^2\sqrt{2\pi}} = \frac{\left(e^{i\xi\frac{b}{2}} - e^{-i\xi\frac{b}{2}}\right)^2}{i^2\xi^2\sqrt{2\pi}}$$

$$\hat{g}(\xi) = \frac{[2i\sin(\xi b/2)]^2}{i^2\xi^2\sqrt{2\pi}} = \frac{4\sin^2(\xi b/2)}{\xi^2\sqrt{2\pi}}$$

$$\boxed{\hat{g}(\xi) = 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin(\xi b/2)}{\xi}\right)^2}$$

Exercises for Fourier Transforms (cont.)

6. (Shifting on the ξ-axis) Show that if f has a Fourier transform $\hat{f}$, then $\hat{f}(\xi-a)$ is the Fourier transform of $e^{iax}f(x)$.

Let $g(x) = e^{iax} f(x)$. Then

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-i\xi x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(\xi-a)x} dx$$

$$\therefore \boxed{\hat{g}(\xi) = \hat{f}(\xi-a)}.$$

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7. Using the result of problem 6, obtain formula G in the table of Fourier transforms from formula A.

Let $f(x) = \begin{cases} 1 & \text{if } -b<x<b, \\ 0 & \text{otherwise}, \end{cases}$ and let $g(x) = e^{iax} f(x)$

$= \begin{cases} e^{iax} & \text{if } -b<x<b, \\ 0 & \text{otherwise}. \end{cases}$ Formula A implies $\hat{f}(\xi) = \sqrt{\frac{2}{\pi}} \frac{\sin(b\xi)}{\xi}$

and applying the result of problem 6 gives

$$\hat{g}(\xi) = \hat{f}(\xi-a) = \boxed{\sqrt{\frac{2}{\pi}} \frac{\sin(b(\xi-a))}{\xi-a}}.$$

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8. Using the result of problem 6, obtain formula H in the table of Fourier transforms from formula B.

Let $f(x) = \begin{cases} 1 & \text{if } c<x<d, \\ 0 & \text{otherwise}, \end{cases}$ and let

Exercises for Fourier Transforms (cont.)

8. (cont.) $g(x) = e^{iax} f(x) = \begin{cases} e^{iax} & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$

By formula B, $\hat{f}(\xi) = \dfrac{e^{-ic\xi} - e^{-id\xi}}{i\xi\sqrt{2\pi}}$. Applying the result of problem 6 gives

$$\hat{g}(\xi) = \hat{f}(\xi - a) = \frac{e^{-ic(\xi-a)} - e^{-id(\xi-a)}}{i(\xi-a)\sqrt{2\pi}} \cdot \frac{i}{i}$$

$$= \boxed{\frac{i\left[e^{ic(a-\xi)} - e^{id(a-\xi)}\right]}{\sqrt{2\pi}\,(a-\xi)}}$$

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9. Verify formula C in the table of Fourier transforms.

Let $g(x) = e^{-a|x|}$ where $a > 0$. Returning to the Fourier transform calculation in problem 3, we easily see that it yields

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a - i\xi} + \frac{1}{a + i\xi}\right] = \frac{2a}{\sqrt{2\pi}\,(a^2 + \xi^2)}.$$

Since g and $\hat{g}$ are absolutely integrable and g is continuous, the Fourier inversion theorem gives

$$\frac{\sqrt{2\pi}}{2a} \cdot e^{-a|t|} = \frac{\sqrt{2\pi}}{2a} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(\eta) e^{i\eta t}\, d\eta$$

$$= \frac{1}{2a} \int_{-\infty}^{\infty} \frac{2a}{\sqrt{2\pi}\,(a^2 + \eta^2)} e^{i\eta t}\, d\eta$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{a^2 + \eta^2} e^{i\eta t}\, d\eta \qquad \text{for all } -\infty < t < \infty.$$

9. (cont.) Setting $t = -\xi$ in this identity, we have

$$\sqrt{\frac{\pi}{2}} \cdot \frac{e^{-a|\xi|}}{a} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{a^2+\eta^2} e^{-i\xi\eta} \, d\eta \qquad \text{for all } -\infty < \xi < \infty.$$

Since the variable η in the definite integral is a dummy variable, the above identity is equivalent to the following statement:

If $\boxed{f(x) = \frac{1}{x^2+a^2}}$ then $\boxed{\hat{f}(\xi) =} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{x^2+a^2} e^{-i\xi x} \, dx = \boxed{\sqrt{\frac{\pi}{2}} \cdot \frac{e^{-a|\xi|}}{a}}$.

This is formula C in the table of Fourier transforms.

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10. Verify formula J in the table of Fourier transforms.

Let $g(x) = \begin{cases} 1 & \text{if } -a<x<a, \\ 0 & \text{otherwise.} \end{cases}$ By formula A in the table of Fourier transforms, $\hat{g}(\xi) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin(a\xi)}{\xi}$. The inversion formula still holds in the form

$$\frac{g(x^+) + g(x^-)}{2} = \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \hat{g}(\xi) e^{i\xi x} \, d\xi$$

for each real number x (see theorem 5C, p. 12, Fourier Transforms by R.R. Goldberg), since g is absolutely integrable and of bounded variation. In particular, away from the discontinuities in g at $\pm a$, we have

$$\begin{cases} 1 & \text{if } |t|<a, \\ 0 & \text{if } |t|>a, \end{cases} = \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \sqrt{\frac{2}{\pi}} \cdot \frac{\sin(a\eta)}{\eta} e^{i\eta t} \, dt,$$

that is,

Exercises for Fourier Transforms (cont.)

10. (cont.) $$\begin{cases} \sqrt{\frac{\pi}{2}} & \text{if } |t| < a, \\ 0 & \text{if } |t| > a, \end{cases} = \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \frac{\sin(a\eta)}{\eta} e^{it\eta}\, d\eta.$$

Setting $t = -\xi$ in this identity yields

$$\begin{cases} \sqrt{\frac{\pi}{2}} & \text{if } |\xi| < a, \\ 0 & \text{if } |\xi| > a, \end{cases} = \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \frac{\sin(a\eta)}{\eta} e^{-i\xi\eta}\, d\eta.$$

Because the variable η in the definite integral is a dummy variable, the above identity is equivalent to the following statement:

If $\boxed{f(x) = \frac{\sin(ax)}{x}}$ where $a > 0$, then

$$\boxed{\hat{f}(\xi) =} \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \frac{\sin(ax)}{x} e^{-i\xi x}\, dx = \boxed{\begin{cases} \sqrt{\frac{\pi}{2}} & \text{if } |\xi| < a, \\ 0 & \text{if } |\xi| > a. \end{cases}}$$

This is formula J in the table of Fourier transforms.

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11. Verify formula D in the table of Fourier transforms.

If A is any subset of the real numbers, let $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise}. \end{cases}$ The function χ_A is called the characteristic function (or indicator function) of the set A. We will make use of an elementary fact about characteristic functions: if A and B are subsets of the real numbers then $\chi_{A\cap B}(x) = \chi_A(x)\chi_B(x)$ for all real x.

We observe that formula B of the table of Fourier

<u>Exercises for Fourier Transforms (cont.)</u>

11. (cont.) transforms, phrased in terms of the characteristic function of the open interval (c,d), says

$$\hat{\chi}_{(c,d)}(\xi) = \frac{e^{-ic\xi} - e^{-id\xi}}{i\xi\sqrt{2\pi}} .$$

In particular, taking $c=0$ and $d=b$, we have

$$(\dagger) \qquad \hat{\chi}_{(0,b)}(\xi) = \frac{1 - e^{-ib\xi}}{i\xi\sqrt{2\pi}} .$$

We claim that

$$(\dagger\dagger) \qquad (\chi_{(0,b)} * \chi_{(0,b)})(x) = \begin{cases} x & \text{if } 0 < x \le b, \\ 2b - x & \text{if } b < x < 2b, \\ 0 & \text{otherwise.} \end{cases}$$

Granting for a moment the validity of $(\dagger\dagger)$, $(\dagger)$ would imply that

$$(\dagger\dagger\dagger) \qquad \widehat{\chi_{(0,b)} * \chi_{(0,b)}}(\xi) = \sqrt{2\pi}\, \hat{\chi}_{(0,b)}(\xi)\, \hat{\chi}_{(0,b)}(\xi)$$

$$= \sqrt{2\pi} \left(\frac{1 - e^{-ib\xi}}{i\xi\sqrt{2\pi}} \right)^2$$

$$= \sqrt{2\pi} \left(\frac{1 - 2e^{-ib\xi} + e^{-2ib\xi}}{-\xi^2(2\pi)} \right)$$

$$= \boxed{\frac{-1 + 2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}} .$$

Comparing $(\dagger\dagger)$ and $(\dagger\dagger\dagger)$, we would see that formula D of the table of Fourier transforms holds.

11. (cont.) It remains only to prove the validity of (††). This we do by a straightforward computation.

$$\chi_{(0,b)} * \chi_{(0,b)}(x) = \int_{-\infty}^{\infty} \chi_{(0,b)}(x-y)\,\chi_{(0,b)}(y)\,dy$$

$$= \int_0^b \chi_{(0,b)}(x-y)\,dy$$

Let $s = x - y$. Then $ds = -dy$.

$$= \int_x^{x-b} \chi_{(0,b)}(s)(-ds)$$

$$= \int_{x-b}^{x} \chi_{(0,b)}(s)\,ds$$

$$= \int_{-\infty}^{\infty} \chi_{(0,b)}(s)\,\chi_{(x-b,x)}(s)\,ds$$

$$= \int_{-\infty}^{\infty} \chi_{(0,b)\cap(x-b,x)}(s)\,ds$$

$= \text{the length of } \{(0,b)\cap(x-b,x)\}$

$$= \begin{cases} x & \text{if } 0 \le x \le b, \\ 2b-x & \text{if } b < x \le 2b, \\ 0 & \text{otherwise.} \end{cases}$$