Mathematics 325
Homework 8

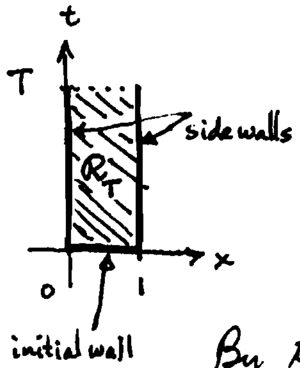
Due Date: ____________________

Name: ________________________

Work exercise 3 on page 44.

3. Consider the diffusion equation $u_t = u_{xx}$ in the interval $(0, 1)$ with $u(0, t) = u(1, t) = 0$ and $u(x, 0) = 1 - x^2$. Note that this initial function does not satisfy the boundary condition at the left end, but that the solution will satisfy it for all $t > 0$.
   (a) Show that $u(x, t) > 0$ at all interior points $0 < x < 1$, $0 < t < \infty$.
   (b) For each $t > 0$, let $\mu(t) =$ the maximum of $u(x, t)$ over $0 \leq x \leq 1$. Show that $\mu(t)$ is a decreasing (i.e., nonincreasing) function of t. (*Hint:* Let the maximum occur at the point $X(t)$, so that $\mu(t) = u(X(t), t)$. Differentiate $\mu(t)$, assuming that $X(t)$ is differentiable.)
   (c) Draw a rough sketch of what you think the solution looks like (u versus x) at a few times. (If you have appropriate software available, compute it.)

(a) Fix $T > 0$. Let



$$R_T = \{(x,t) : 0 < x < 1 ,\ 0 < t < T\}$$

m_T = minimum of $u(x,t)$ on the sidewalls and initial wall of R_T,
M_T = maximum of $u(x,t)$ " " " " " " " ".

By the strong maximum/minimum principle

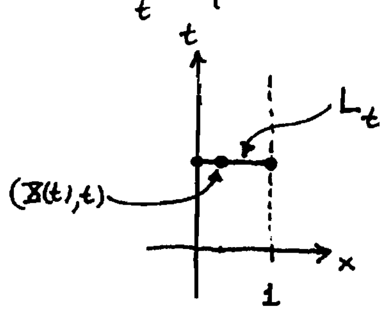
$$m_T \le u(x,t) \le M_T$$

for all $(x,t) \in R_T$, with equality only if u is a constant function on $\overline{R_T}$. However $u(x,0) = 1 - x^2 \not\equiv$ constant for $0 \le x \le 1$. Therefore $m_T < u(x,t) < M_T$ for all $(x,t) \in R_T$. It is clear that $m_T = 0$ and $M_T = 1$, and hence

$$(*) \qquad 0 < u(x,t) < 1$$

for all $(x,t) \in R_T$. Because $T > 0$ is arbitrary, $(*)$ holds for all points (x,t) such that $0 < x < 1$ and $0 < t < \infty$.

(b) Let $\bar{X}(t)$ denote the number in $[0,1]$ such that $\mu(t) = u(\bar{X}(t), t)$, i.e. the maximum of $u(x,t)$ on the horizontal line segment $L_t = \{(x,t) : 0 \le x \le 1\}$ occurs at the point $(\bar{X}(t), t)$.



(If the maximum of $u(x,t)$ on L_t occurs at more than one point, then take $(\bar{X}(t), t)$ as the left-most point on L_t where the maximum of $u(x,t)$ occurs.)

By the chain rule applied to $\mu(t) = u(\bar{X}(t), t)$, we have

$$\frac{d\mu}{dt} = \frac{\partial u}{\partial \bar{X}} \cdot \frac{d\bar{X}}{dt} + \frac{\partial u}{\partial t} \cdot \overbrace{\frac{dt}{dt}}^{1}$$

$$= \frac{\partial u}{\partial \bar{X}} \cdot \frac{d\bar{X}}{dt} + \frac{\partial^2 u}{\partial x^2}$$

since u is a solution to $u_t - u_{xx} = 0$. But $\frac{\partial u}{\partial x}(\bar{X}(t), t) = 0$ and $\frac{\partial^2 u}{\partial x^2}(\bar{X}(t), t) \leq 0$ since the maximum of u on L_t occurs at $(\bar{X}(t), t)$. Consequently

$$\frac{d\mu}{dt} = 0 \cdot \frac{d\bar{X}}{dt} + \frac{\partial^2 u}{\partial x^2} \leq 0 ;$$

i.e. μ is a nonincreasing function of t.

(c)