Mathematics 315 | Midterm Exam Spring 2006 | Name:____________________

1.(25 pts.) Let f be a bounded real function on $[a,b]$ and let α be a real function on $[a,b]$.

(a) If α is increasing on $[a,b]$, define the phrase "f is Riemann-Stieltjes integrable with respect to α on $[a,b]$".

(b) Define what it means for α to be of bounded variation on $[a,b]$.

(c) State Jordan's theorem relating functions of bounded variation and increasing functions.

(d) State the definition of the Riemann-Stieltjes integral of f with respect to α on $[a,b]$ if f is continuous on $[a,b]$ and α is of bounded variation on $[a,b]$.

(e) If f is continuous on $[0,1]$ and $\alpha(x)=\sum_{n=2}^{\infty}\frac{1}{n^2}H\left(x-\frac{1}{n}\right)$, write, without proof, a formula for the value of $\int_0^1 f d\alpha$. (Here H denotes the unit Heaviside step function.)

(f) If f is Riemann integrable on $[0,1]$ and α is differentiable with α' Riemann integrable on $[0,1]$, write, without proof, a formula for the value of $\int_0^1 f d\alpha$.

2.(25 pts.) Consider the vector space $C[a,b]$ of all continuous real functions on the interval $[a,b]$.

(a) Define the phrase "N is a norm on $C[a,b]$".

(b) If N is a norm on $C[a,b]$, define the phrase "N is a Banach space norm on $C[a,b]$".

(c) Give, without proof, an example of a norm on $C[a,b]$ which is **not** a Banach space norm.

(d) Give, without proof, an example of a norm on $C[a,b]$ which **is** a Banach space norm.

(e) Define the phrase "Λ is a bounded linear functional on the normed linear space $(C[a,b],N)$".

(f) State, without proof, the Riesz Representation Theorem characterizing the bounded linear functionals on the space $C[a,b]$. (Be sure to clearly state the Banach space norm that is being used on $C[a,b]$.)

(g) Show that $\Lambda(f)=f(0)-3f(1/2)+2\int_0^1 f(x)x\,dx$ defines a bounded linear functional on $C[0,1]$ equipped with an appropriate Banach space norm. Then find a function α corresponding to Λ guaranteed by the Riesz Representation Theorem.

3.(25 pts.) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real functions on $[a,b]$, and let f be a real function on $[a,b]$.

(a) Define the phrase "$\{f_n\}_{n=1}^{\infty}$ converges to f pointwise on $[a,b]$".

(b) Define the phrase "$\{f_n\}_{n=1}^{\infty}$ converges to f uniformly on $[a,b]$".

(c) Give, without proof, an example of a sequence of functions $\{f_n\}_{n=1}^{\infty}$ which is pointwise convergent but not uniformly convergent on $[0,1]$.

(d) State the Stone-Weierstrass approximation theorem.

(e) Briefly sketch the steps you would take in showing that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f\left(n\pi\sqrt{2}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\,dt$$

for all continuous, complex, 2π-periodic functions f.

4.(25 pts.) Let f be a 2π – periodic real function which is Riemann integrable on $[-\pi, \pi]$.

(a) Define the Fourier coefficients and Fourier partial sums of f.

(b) Define the Dirichlet and Fejer kernels and tell how they are related to the Fourier partial sums of f.

(c) State the Dirichlet-Jordan theorem for convergence of the Fourier partial sums of f.

(d) State Fejer's theorem on uniform convergence of the arithmetic means of the Fourier partial sums of f.