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#2

(a) \[ L(u + v) = (u + v)_x + (u + v)_y = u_x + v_x + xu_y + xv_y \]
    \[ = u_x + xu_y + v_x + xv_y = L(u) + L(v) \]
    \[ L(ku) = (ku)_x + (ku)_y = ku_x + xku_y = kL(u) \]
    \( \{ \) Linear \( \} \)

(b) \[ L(u + v) = (u + v)_x + (u + v)(u + v)_y = u_x + v_x + (u + v)(u_y + v_y) \]
    \[ = u_x + uuv_y + v_x + vv_y + u_y + v_y \]
    \[ = L(u) + L(v) + uyy + vvy \]
    \( \text{not always zero} \) \( \) Nonlinear \( \}

(c) \[ L(u + v) = (u + v)_x + (u + v)^2_y = u_x + v_x + (u_y + v_y)^2 \]
    \[ = u_x + u_y^2 + v_x + v_y^2 + 2uv_y \]
    \[ = L(u) + L(v) + 2u_yv_y \]
    \( \text{not always zero} \) \( \) Nonlinear \( \}

(d) \[ L(u + v) = (u + v)_x + (u + v)_y + 1 \]
    \[ = u_x + u_y + 1 + v_x + v_y + 1 - 1 \]
    \[ = L(u) + L(v) - \] \( \text{not zero} \) \( \) Nonlinear \( \}

(e) \[ L(u + v) = \sqrt{1 + x^2 \cos(y)(u + v)_x + (u + v)_y} - [\arctan(x/y)](u + v) \]
    \[ = \sqrt{1 + x^2 \cos(y)u_x + u_y} - [\arctan(x/y)]u + \sqrt{1 + x^2 \cos(y)v_x + v_y} - [\arctan(x/y)]v \]
    \[ = L(u) + L(v) \]
#2 (c) (cont.) \[ L(ku) = \sqrt{1 + u_x^2} \cos(y)(ku)_x + (ku)_{yy} - \left[ \arctan \left( \frac{x}{y} \right) \right](ku) \]
\[ = k \left[ \sqrt{1 + u_x^2} \cos(y)u_x + u_{yy} - \left[ \arctan \left( \frac{x}{y} \right) \right]u \right] \]
\[ = k L(u) \]
\[ \therefore \text{L is linear.} \]

#3 (a) \[ u_t - u_{xx} + 1 = 0 \] Second order, linear, inhomogeneous
(b) \[ u_t - u_{xx} + xu = 0 \] Second order, linear, homogeneous
(c) \[ u_t - u_{xx} + \left( uu_x \right)_x = 0 \] Third order, nonlinear
(d) \[ u_{tt} - u_{xx} + \left( u_x^2 \right)_x = 0 \] Second order, linear, inhomogeneous
(e) \[ i u_t - u_{xx} + \frac{1}{k} u = 0 \] Second order, linear, homogeneous
(f) \[ u_x (1 + u_x)^{\frac{1}{2}} + u_y (1 + u_y)^{\frac{1}{2}} = 0 \] First order, nonlinear

(g) \[ u_x + e u_y = 0 \] First order, linear, homogeneous
(h) \[ u_t + u_{xxxx} + \sqrt{1 + u_x} = 0 \] Fourth order, nonlinear

#5 (a) \[ V_1 = \{ [a, b, c] \in \mathbb{R}^3 : b = 0 \} \] is a vector space because it is a subset of the vector space \( \mathbb{R}^3 \) which is closed under the operations of addition of vectors and scalar multiplication:
\[ [a_1, 0, c_1] + [a_2, 0, c_2] = [a_1 + a_2, 0, c_1 + c_2] \in V_1, \]
\[ k[a, 0, c] = [ka, 0, kc] \in V_1. \]
$\#5 \; (b) \quad V_2 = \ \{ \ [a, b, c] \in \mathbb{R}^3 : \ b = 1 \ \} \text{ is not a vector space because it is not closed under addition of vectors:} $

\[ [a_1, 1, c_1] + [a_2, 1, c_2] = [a_1 + a_2, 1, c_1 + c_2] \notin V_2. \]

(c) \quad V_3 = \ \{ \ [a, b, c] \in \mathbb{R}^3 : \ ab = 0 \ \} \text{ is not a vector space because it is not closed under addition of vectors:} $

\[ [0, 1, c_1] + [1, 0, c_2] = [1, 1, c_1 + c_2] \notin V_3. \]

(d) \quad V_4 = \ \{ \ k_1 [1, 1, 0] + k_2 [2, 0, 1] \in \mathbb{R}^3 : \ k_1, k_2 \in \mathbb{R} \ \} \text{ is a vector space because it is a subset of the vector space } \mathbb{R}^3 \text{ which is closed under the operations of addition of vectors and scalar multiplication:} $

\[ k_1 [1, 1, 0] + k_2 [2, 0, 1] + k_1' [1, 1, 0] + k_2' [2, 0, 1] = (k_1 + k_1') [1, 1, 0] + (k_2 + k_2') [2, 0, 1], \]

\[ k (k_1 [1, 1, 0] + k_2 [2, 0, 1]) = kk_1 [1, 1, 0] + kk_2 [2, 0, 1] \in V_4. \]

(e) \quad V_5 = \ \{ \ [a, b, c] \in \mathbb{R}^3 : \ c - a = 2b \ \} \text{ is a vector space because it is a subset of the vector space } \mathbb{R}^3 \text{ which is closed under the operations of addition of vectors and scalar multiplication:} $

\[ [a_1, \frac{c_1 - a_1}{2}, c_1] + [a_2, \frac{c_2 - a_2}{2}, c_2] = [a_1 + a_2, \frac{c_1 + c_2 - (a_1 + a_2)}{2}, c_1 + c_2] \in V_5, \]

\[ k [a, \frac{c - a}{2}, c] = [ka, \frac{kc - ka}{2}, kc] \in V_5. \]
#7 Suppose that \( c_1(1+x) + c_2(1-x) + c_3(1+x+x^2) = 0 \) for some constants \( c_1, c_2, c_3 \). Then
\[
(c_1 + c_2 + c_3) + (c_1 - c_2 + c_3)x + c_3x^2 = 0,
\]
and so \( c_1 + c_2 + c_3 = c_1 - c_2 + c_3 = c_3 = 0 \). (The only polynomial equal to zero for arbitrary \( x \) is the one whose coefficients all vanish.) It follows that \( c_3 = 0 \) and \( c_1+c_2 = 0 = c_1-c_2 \). From the last string of equalities, we have (upon addition of equations) that \( 2c_1 = 0 \). Hence \( c_1 = 0 \) and consequently \( c_2 = 0 \) as well. We have shown that the only choice of constants \( c_1, c_2, \) and \( c_3 \) such that
\[
c_1(1+x) + c_2(1-x) + c_3(1+x+x^2) = 0
\]
is the trivial choice \( c_1 = c_2 = c_3 = 0 \). Therefore \( 1+x \), \( 1-x \), and \( 1+x+x^2 \) are linearly independent functions.

#9 \( \mathcal{V} = \{ c_1 + c_2 \sin^2(x) + c_3 \cos^2(x) : c_1, c_2, c_3 \in \mathbb{R} \} \) is a vector space because it is a subset of the vector space of all functions on \( \mathbb{R} \) which is closed under addition of functions and scalar multiplication:
\[
\begin{align*}
&c_1 + c_2 \sin^2(x) + c_3 \cos^2(x) + c_1' + c_2' \sin^2(x) + c_3' \cos^2(x) \\
&= (c_1+c_1') + (c_2+c_2')\sin^2(x) + (c_3+c_3')\cos^2(x) \in \mathcal{V}, \\
k[c_1 + c_2 \sin^2(x) + c_3 \cos^2(x)] = kc_1 + kc_2 \sin^2(x) + kc_3 \cos^2(x) \in \mathcal{V}.
\end{align*}
\]
#9 (cont.) Using the Pythagorean identity \( \sin^2(x) + \cos^2(x) = 1 \), we see that
\[
c_1 + c_2 \sin^2(x) + c_3 \cos^2(x) = (c_1 + c_3) + (c_2 - c_3) \sin^2(x).
\]
Because the functions 1 and \( \sin^2(x) \) are linearly independent, it follows that \( V \) is two-dimensional and \( \{1, \sin^2(x)\} \) is a basis for \( V \).

#10 Let \( S = \{ u : u'' - 3u'' + 4u = 0 \} \), and let \( u_1, u_2 \in S \), \( k \in \mathbb{R} \).

Then \( u_1'' - 3u_1'' + 4u_1 = 0 \) and \( u_2'' - 3u_2'' + 4u_2 = 0 \). Adding equations and using linearity of differentiation, we find
\[
(u_1 + u_2)'' - 3(u_1 + u_2)'' + 4(u_1 + u_2) = 0.
\]
That is, \( u_1 + u_2 \in S \). Likewise,
\[
(ku_1)'' - 3(ku_1)'' + 4(ku_1) = k(u_1'' - 3u_1'' + 4u_1) = 0
\]
so \( ku_1 \in S \). Thus \( S \) is a subset of the vector space of thrice differentiable functions which is closed under addition of functions and scalar multiplication, and consequently \( S \) is a vector space.

To find a basis for \( S \), consider \( u = e^{mx} \) where \( m \) is a constant to be determined so that \( u = e^{mx} \) belongs to \( S \). Then
\[
m^3 e^{mx} - 3m^2 e^{mx} + 4e^{mx} = u'' - 3u'' + 4u = 0
\]
so (dividing through by \( e^{mx} \)) we have \( m^3 - 3m^2 + 4 = 0 \). But
Sec. 1.1, pp. 4-5.

#10 (cont.) This factors as \((m-2)(m^2-m-2) = 0\) or \((m-2)(m-2)(m+1) = 0\). Thus \(m = -1\) or \(m = 2\) (double root).

It follows from elementary ODE theory that
\[
u = (c_1 + c_2 x)e^{2x} + c_3 e^{-x}
\]
is the general solution of \(u'' - 3u'' + 4u = 0\) (as one can easily check).

Thus \(\mathcal{S}\) is three-dimensional with basis \(\{e^{2x}, xe^{2x}, e^{-x}\}\).

#11 Let \(f\) and \(g\) be differentiable functions of one variable and let \(u(x,y) = f(x)g(y)\). Then \(u_x = f'(x)g(y)\), \(u_y = f(x)g'(y)\), and \(u_{xy} = f(x)g'(y)\) so
\[
u_{xx} - u_x u_y = f(x)g(y)f'(x)g'(y) - f'(x)g'(y)f(x)g(y) = 0;
\]
that is, \(u(x,y) = f(x)g(y)\) is a solution of the PDE \(u_{xx} = u_x u_y\).

#12 Let \(n > 0\) and consider \(u_n(x,y) = \sin(nx) \sinh(ny)\). Then
\[
(u_n)_x = n \cos(nx) \sinh(ny), \quad (u_n)_{xx} = -n^2 \sin(nx) \sinh(ny),
\]
\[
(u_n)_y = n \sin(nx) \cosh(ny), \quad (u_n)_{yy} = n^2 \sin(nx) \sinh(ny).\]
Consequently,
\[
(u_n)_{xx} + (u_n)_{yy} = -n^2 \sin(nx) \sinh(ny) + n^2 \sin(nx) \sinh(ny) = 0.
\]