Sec. 1.4, pp. 24–25.

#1. By trial and error, find a solution of the diffusion equation $u_t = u_{xx}$ with the initial condition $u(x,0) = x^2$.

We assume a solution of polynomial form:

$$u(x,t) = \sum_{i,j=0}^{N} c_{ij} x^i t^j$$

where $c_{0,0}, c_{0,1}, c_{1,0}, \dots, c_{N,N}$ are constants.

The initial condition implies

$$x^2 = u(x,0) = \sum_{i,j=0}^{N} c_{ij} x^i 0^j = \sum_{i=0}^{N} c_{i0} x^i = c_{0,0} + c_{1,0} x + c_{2,0} x^2 + \dots + c_{N,0} x^N$$

and hence $\boxed{c_{0,0} = c_{1,0} = c_{3,0} = \dots = c_{N,0} = 0}$ and $\boxed{c_{2,0} = 1}$. Thus

$$u(x,t) = x^2 + \sum_{i=0}^{N} \sum_{j=1}^{N} c_{ij} x^i t^j .$$

Differentiating we have

$$\frac{\partial u}{\partial t} = \sum_{i=0}^{N} \sum_{j=1}^{N} j c_{ij} x^i t^{j-1} \overset{\text{Let } j' = j-1}{=} \sum_{i=0}^{N} \sum_{j'=0}^{N-1} (j'+1) c_{i,j'+1} x^i t^{j'},$$

$$\frac{\partial^2 u}{\partial x^2} = 2 + \sum_{j=1}^{N} \sum_{i=0}^{N} i(i-1) c_{ij} x^{i-2} t^j \underset{\text{Let } i' = i-2}{=} 2 + \sum_{j=1}^{N} \sum_{i'=0}^{N-2} (i'+1)(i'+2) c_{i'+2,j} x^{i'} t^j .$$

Substituting these expressions into $u_t = u_{xx}$ yields the recurrence relation:

(1) $$(j+1) c_{i,j+1} = (i+1)(i+2) c_{i+2,j} \quad \text{for } 0 \le i \le N-2 \text{ and } 1 \le j \le N-1;$$

as well as the additional relations:

(2) $$\boxed{c_{0,1} = 2 \quad \text{and} \quad c_{i,1} = 0 \quad \text{for } 1 \le i \le N};$$

(3) $$c_{N-1,j+1} = c_{N,j+1} = 0 \quad \text{for } 1 \le j \le N-1;$$

(4) $$c_{i+2,N} = 0 \quad \text{for } 0 \le i \le N-2.$$

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#1. (cont.) Using ② and ① (with $j=1$) we see that $c_{i,2}=0$ for $0\le i\le N-2$. But then ③ (with $j=1$) implies $c_{N-1,2}=c_{N,2}=0$ as well, and hence

$$c_{i,2}=0 \quad \text{for } 0\le i\le N.$$

Using the same argument, we find likewise that

$$c_{i,3}=0 \quad \text{for } 0\le i\le N.$$

An easy induction yields

$$\boxed{c_{i,j}=0 \quad \text{for } 0\le i\le N \text{ and } 2\le j\le N.}$$

From the boxed equations we see that $c_{2,0}=1$, $c_{0,1}=2$, and $c_{i,j}=0$ for all other i and j. Consequently

$$\boxed{u(x,t)=\sum_{i,j=0}^{N} c_{ij}x^i t^j = c_{0,1}x^0t^1+c_{2,0}x^2t^0 = 2t+x^2}$$

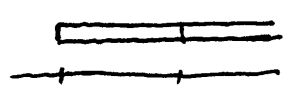
is a solution to $u_t=u_{xx}$ satisfying $u(x,0)=x^2$.

#2. (a) *Show that the temperature of a metal rod, insulated at the end $x=0$, satisfies the boundary condition $\partial u/\partial n=0$. (Use Fourier's law.)*

(b) *Do the same for the diffusion of gas along a tube that is closed off at the end $x=0$. (Use Fick's law.)*

(c) *Show that the three-dimensional version of (a) (insulated solid) or (b) (impermeable container) leads to the boundary condition $\partial u/\partial n=0$.*

(a) Let $u(x,t)$ denote the temperature of the metal rod at position x and time t, and let $H(t)$ denote the heat energy contained in the rod between $x=0$ and $x=\Delta x$.



$x=0$ $\quad$ $x=\Delta x$

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#2 (cont.) Then

$$H(t) = \int_0^{\Delta x} c\rho u(\xi,t)\,d\xi$$

where c = the specific heat of the material of the rod

and ρ = the linear density " " " " " " .

The rate of change of heat energy,

$$\frac{dH}{dt} = \frac{d}{dt}\int_0^{\Delta x} c\rho u(\xi,t)\,d\xi = \int_0^{\Delta x} c\rho u_t(\xi,t)\,d\xi ,$$

is equal to the heat flux across the boundaries $x=0$ and $x=\Delta x$ of this segment of rod. At $x=0$ there is no heat energy flux because that end is insulated. At $x=\Delta x$ the flux of heat energy is proportional to the temperature gradient (Fourier's law). Thus

$$\frac{dH}{dt} = -K u_x(\Delta x, t) + 0$$

Equating the two expressions for dH/dt and letting $\Delta x \to 0$ yields

$$0 = \lim_{\Delta x\to 0}\int_0^{\Delta x} c\rho u_t(\xi,t)\,d\xi = \lim_{\Delta x\to 0} -K u_x(\Delta x,t) = -K u_x(0,t).$$

That is, $u_x(0,t) = 0$ for $t \geq 0$.

(b) Using the same notation as in example 4 of Sec. 1.3, the mass in the segment of pipe from $x=0$ to $x=\Delta x$ at time t,

$$M = \int_0^{\Delta x} u(\xi,t)\,d\xi ,$$

#2 (cont.) has a time rate of change

$$\frac{dM}{dt} = \frac{d}{dt}\int_0^{\Delta x} u(\xi,t)\,d\xi = \int_0^{\Delta x} u_t(\xi,t)\,d\xi .$$

This rate of change of mass results from a flux of particles only at the end $x = \Delta x$ of the segment of pipe since the pipe is closed off at the end $x=0$. By Fick's law

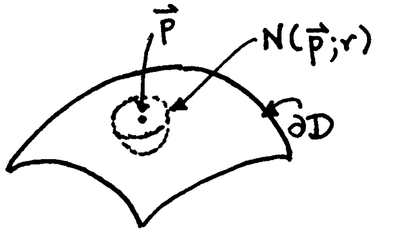
$$\frac{dM}{dt} = k u_x(\Delta x, t).$$

Equating these two expressions for dM/dt and letting $\Delta x \to 0$ yields

$$0 = \lim_{\Delta x \to 0} \int_0^{\Delta x} u_t(\xi,t)\,d\xi = \lim_{\Delta x \to 0} k u_x(\Delta x, t) = k u_x(0,t) .$$

That is, $u_x(0,t) = 0$ for all $t \geq 0$.

(c) We consider only the case of heat flow, the diffusion process being completely analogous. Let $\vec{P}$ be a point on ∂D,



the boundary of D, and let $N(\vec{P};r)$ denote those points in D which are at most a distance r from $\vec{P}$. Using the same notation as in example 5 of Sec. 1.3, the heat energy in $N(\vec{P};r)$ at time t,

$$H(t) = \iiint_{N(\vec{P};r)} c\rho u(\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta$$

#2 (cont.) where

$c =$ the specific heat of the material in D,
$\rho =$ the density " " " " ",

has a rate of change

$$\frac{dH}{dt} = \frac{d}{dt}\iiint_{N(\vec{p};r)} c\rho u(\xi,\eta,\zeta,t)\,d\xi\,d\eta\,d\zeta = \iiint_{N(\vec{p};r)} c\rho u_t(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta.$$

Heat energy in $N(\vec{p};r)$ changes only due to heat flux through the boundary. For that portion of the boundary $\partial N(p;r)$ of $N(\vec{p};r)$ which is on the boundary ∂D of D, there is no heat flux because it is insulated. For the portion of $\partial N(\vec{p};r)$ which is in (the interior of) D, Fourier's law says that the heat flows from hot to cold regions proportionately to the temperature gradient. Therefore

$$\frac{dH}{dt} = \iint_{D \cap \partial N(\vec{p};r)} K\left(\vec{n}(\xi,\eta,\zeta)\cdot\nabla u(\xi,\eta,\zeta)\right)dS$$

where $K =$ the heat conducdivity of the material in D

and $\vec{n}(\xi,\eta,\zeta) =$ the outward-pointing normal vector to $\partial N(\vec{p};r)$ at the point (ξ,η,ζ).

Equating these two expressions for dH/dt, dividing by

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#2 (cont.) $S(D \cap \partial N(\vec{p};r))$, the surface area of $D \cap N(\vec{p};r)$, and letting $r \to 0^+$ gives

$$0 = \lim_{r\to 0^+} \frac{1}{S(D\cap\partial N(\vec{p};r))} \iiint_{N(\vec{p};r)} c\rho u_t(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta$$

$$= \lim_{r\to 0^+} \frac{1}{S(D\cap\partial N(\vec{p};r))} \iint_{D\cap\partial N(\vec{p};r)} K\left(\vec{n}(\xi,\eta,\zeta)\cdot\nabla u(\xi,\eta,\zeta)\right)dS$$

$$= -K\left(\vec{n}(\vec{p})\cdot\nabla u(\vec{p})\right)$$

That is, $\frac{\partial u}{\partial n} = \vec{n}\cdot\nabla u = 0$ at $\vec{p}$ for all $t \geq 0$.

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A homogeneous body occupying the solid region D is completely insulated. Its initial temperature is $f(\vec{x})$. Find the steady-state temperature that it reaches after a long time.

Solution: The heat energy $H(t)$ contained in region D at time t is

$$H(t) = \iiint_D c\rho\, u(\vec{x},t)\, dV$$

where $c\rho$ = constant (due to homogeneity). Then

$$\begin{aligned}
\frac{dH}{dt} &= \iiint_D c\rho\, u_t(\vec{x},t)\, dV \\
&= \iiint_D \kappa \nabla^2 u\, dV \qquad \text{(by the heat equation)} \\
&= \kappa \iiint_D \nabla\cdot(\nabla u)\, dV \\
&= \kappa \iint_{\partial D} \nabla u\cdot\vec{n}\, dS \qquad \text{(by the divergence theorem)} \\
&= 0
\end{aligned}$$

since "complete insulation" implies $\frac{\partial u}{\partial n} = \nabla u\cdot\vec{n} = 0$ at all points of ∂D. Therefore

$$\iiint_D u(\vec{x},t)\, dV = \frac{H(t)}{c\rho} = \text{constant},$$

and hence the heat energy in D is "conserved". In particular, for any time t, no matter how large,

$$\iiint_D u(\vec{x},t)\,dV = \iiint_D u(\vec{x},0)\,dV = \iiint_D f(\vec{x})\,dV. \qquad (*)$$

Assuming that the temperature at steady-state is constant throughout D, i.e.

$$u(\vec{x}) = \lim_{t\to\infty} u(\vec{x},t) = \text{constant} = U \quad (\text{say}),$$

then

$$U\,\mathrm{vol}(D) = \iiint_D u(\vec{x})\,dV = \lim_{t\to\infty}\iiint_D u(\vec{x},t)\,dV \overset{\text{by } (*) \text{ above}}{=} \iiint_D f(\vec{x})\,dV,$$

and hence

$$\boxed{U = \iiint_D f(\vec{x})\,dV \Big/ \mathrm{vol}(D)}\,.$$

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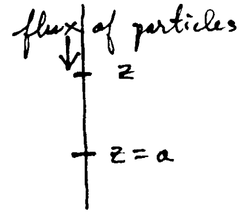
#4 In exercise 4 of Sec 1.3, find the boundary condition on an impermeable plane $z = a$.

Using the same notation as in exercise 4 of Sec. 1.3, the total amount of particles between a and z at a fixed time $t > 0$ is

$$M = \int_a^z u(\xi,t)\,d\xi ,$$



the time rate of change of the amount of particles is

$$\frac{dM}{dt} = \int_a^z u_t(\xi,t)\,d\xi ,$$

and the loss or gain of particles occurs only at the boundary z (the plane at $z = a$ is impermeable) so

$$\frac{dM}{dt} = \text{flux of particles at } z$$

$$= \underbrace{k u_z(z,t)}_{\text{by Fick's law}} + \underbrace{V u(z,t)}_{\text{flux due to downward drift}} .$$

Equating these two expressions for dM/dt, we have (upon letting $z \to a^+$) that

$$0 = \lim_{z \to a^+} \int_a^z u_t(\xi,t)\,dt = \lim_{z \to a^+} \left(k u_z(z,t) + V u(z,t) \right)$$

or

$$\boxed{0 = k u_z(a,t) + V u(a,t)} .$$

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#6 In linearized gas dynamics (sound), verify the following.

(a) If curl $\vec{v} = \vec{0}$ at $t=0$, then curl $\vec{v} = \vec{0}$ at all later times.

(b) Each component of $\vec{v}$ and ρ satisfies the wave equation.

We will make extensive use of the equations

$$(*) \qquad \frac{\partial \vec{v}}{\partial t} + \frac{c_0^2}{\rho_0} \nabla \rho = \vec{0}$$

$$(**) \qquad \frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \vec{v} = 0$$

governing the air density $\rho = \rho(x,y,z,t)$ and velocity field $\vec{v} = \vec{v}(x,y,z,t)$ of the sound disturbance. (See the SOUND illustration of Sec. 1.4. There the notations grad ρ and div $\vec{v}$ are used for the gradient of ρ ($\nabla\rho$) and the divergence of $\vec{v}$ ($\vec{\nabla}\cdot\vec{v}$), respectively.)

(a) Suppose curl $\vec{v} = \overbrace{\nabla \times \vec{v}}^{\text{alternate notation for curl of } \vec{v}} = \vec{0}$ at $t = 0$. Then

$$\begin{aligned}
\frac{\partial}{\partial t}(\nabla \times \vec{v}) &= \nabla \times \frac{\partial \vec{v}}{\partial t} \\
&= \nabla \times \left(-\frac{c_0^2}{\rho_0} \nabla \rho\right) && \text{from } (*) \\
&= -\frac{c_0^2}{\rho_0} \nabla \times \nabla \rho \\
&= \vec{0}
\end{aligned}$$

because the curl of any gradient is zero. (For a quick

#6 (cont.) proof of this, note that $\nabla\rho = \vec{\imath}\rho_x + \vec{\jmath}\rho_y + \vec{k}\rho_z$ and

hence
$$\nabla\times\nabla\rho = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \rho_x & \rho_y & \rho_z \end{vmatrix} = \vec{\imath}\,(\overbrace{\rho_{zy}-\rho_{yz}}^{0}) + \vec{\jmath}\,(\overbrace{\rho_{xz}-\rho_{zx}}^{0}) + \vec{k}\,(\overbrace{\rho_{yx}-\rho_{xy}}^{0})$$
$$= \vec{0}.\ )$$

Since the time rate of change of $\nabla\times\vec{v}$ is the zero-vector, and since $\nabla\times\vec{v} = \vec{0}$ when $t=0$, it follows that $\nabla\times\vec{v} = \vec{0}$ for all $t \geq 0$.

(b)
$$\begin{aligned}
\frac{\partial^2\rho}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial\rho}{\partial t}\right) \\
&= \frac{\partial}{\partial t}\left(-\rho_0\nabla\cdot\vec{v}\right) && \text{from } (**) \\
&= -\rho_0\nabla\cdot\frac{\partial\vec{v}}{\partial t} \\
&= -\rho_0\nabla\cdot\left(-\frac{c_0^2}{\rho_0}\nabla\rho\right) && \text{from } (*) \\
&= c_0^2\nabla^2\rho
\end{aligned}$$

Thus the density satisfies the wave equation.

$$\begin{aligned}
\left(\frac{\partial^2 v_1}{\partial t^2}, \frac{\partial^2 v_2}{\partial t^2}, \frac{\partial^2 v_3}{\partial t^2}\right) &= \frac{\partial^2\vec{v}}{\partial t^2} \\
&= \frac{\partial}{\partial t}\left(\frac{\partial\vec{v}}{\partial t}\right) \\
&= \frac{\partial}{\partial t}\left(\frac{-c_0^2}{\rho_0}\nabla\rho\right) && \text{from } (*)
\end{aligned}$$

#6 (cont.)

$$= -\frac{c_0^2}{\rho_0}\nabla\left(\frac{\partial \rho}{\partial t}\right)$$

$$= -\frac{c_0^2}{\rho_0}\nabla\left(-\rho_0 \nabla\cdot\vec{v}\right) \qquad \text{from } (**)$$

$$= c_0^2 \nabla(\nabla\cdot\vec{v})$$

The above vector identity is equivalent to the three scalar equations

$$\square \quad \begin{cases} \dfrac{\partial^2 v_1}{\partial t^2} = c_0^2\left(\dfrac{\partial^2 v_1}{\partial x^2} + \dfrac{\partial^2 v_2}{\partial x \partial y} + \dfrac{\partial^2 v_3}{\partial x \partial z}\right), \\[2ex] \dfrac{\partial^2 v_2}{\partial t^2} = c_0^2\left(\dfrac{\partial^2 v_1}{\partial y \partial x} + \dfrac{\partial^2 v_2}{\partial y^2} + \dfrac{\partial^2 v_3}{\partial y \partial z}\right), \\[2ex] \dfrac{\partial^2 v_3}{\partial t^2} = c_0^2\left(\dfrac{\partial^2 v_1}{\partial z \partial x} + \dfrac{\partial^2 v_2}{\partial z \partial y} + \dfrac{\partial^2 v_3}{\partial z^2}\right). \end{cases}$$

From part (a), we have for all $t \geq 0$ and all (x,y,z) that

$$(\dagger) \qquad \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} = \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0.$$

Therefore $\dfrac{\partial}{\partial y}\left(\dfrac{\partial v_2}{\partial x} - \dfrac{\partial v_1}{\partial y}\right) = 0 = \dfrac{\partial}{\partial z}\left(\dfrac{\partial v_1}{\partial z} - \dfrac{\partial v_3}{\partial x}\right)$

and hence

$$\frac{\partial^2 v_2}{\partial y \partial x} + \frac{\partial^2 v_3}{\partial z \partial x} = \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}.$$

Substituting in the first equation of the system $\square$ gives

$$\frac{\partial^2 v_1}{\partial t^2} = c_0^2\left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}\right) = c_0^2 \nabla^2 v_1$$

i.e. the first component of $\vec{v}$ satisfies the wave equation.

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#6 (cont.) Similarly arguments using (†) and □ yield

$$\frac{\partial^2 v_2}{\partial t^2} = c_0^2\left(\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} + \frac{\partial^2 v_2}{\partial z^2}\right) = c_0^2 \nabla^2 v_2 ,$$

$$\frac{\partial^2 v_3}{\partial t^2} = c_0^2\left(\frac{\partial^2 v_3}{\partial x^2} + \frac{\partial^2 v_3}{\partial y^2} + \frac{\partial^2 v_3}{\partial z^2}\right) = c_0^2 \nabla^2 v_3 .$$