#1. Consider the problem

D.E. $$\frac{d^2u}{dx^2} + u = 0 \quad \text{for} \quad 0 < x < L,$$

B.C.'s $$u(0) = u(L) = 0.$$

Clearly the function $u(x) \equiv 0$ is a solution. Is this solution unique or not? Does the answer depend on L?

The general solution to the D.E. is $u(x) = c_1 \cos(x) + c_2 \sin(x)$ where c_1 and c_2 are arbitrary constants. The condition $u(0) = 0$ implies $c_1 = 0$, i.e. $u(x) = c_2 \sin(x)$. The condition $u(L) = 0$ implies $c_2 \sin(L) = 0$. Since the zeros of the sine function occur at the integer multiples of π, it follows that the solution to the problem is unique if and only if L is not a positive integer multiple of π.

#2. Consider the problem

$$(*) \quad \begin{cases} u''(x) + u'(x) = f(x) & \text{for } 0 < x < \ell, \\ u'(0) = u(0) = \frac{1}{2}\left[u'(\ell) + u(\ell)\right], \end{cases}$$

with $f = f(x)$ a given function.

(a) Is the solution unique? Explain.

(b) Does a solution necessarily exist, or is there a condition that f must satisfy for existence? Explain.

#2 (cont.) (a) Suppose u_1 and u_2 are solutions to (*). Then $v = u_1 - u_2$ solves the B.V.P.

$$① \quad v'' + v' = 0,$$

$$② \quad v'(0) - v(0) = 0,$$

$$③ \quad v'(\ell) + v(\ell) - 2v(0) = 0.$$

The general solution of ① is

$$(\dagger) \qquad v(x) = c_1 + c_2 e^{-x}$$

where c_1 and c_2 are arbitrary constants. Hence

$$(\dagger\dagger) \qquad v'(x) = -c_2 e^{-x}.$$

By $(\dagger)$, $(\dagger\dagger)$, and ②, $-c_2 - (c_1 + c_2) = 0$, that is, $c_1 = -2c_2$, and thus

$$(\dagger\dagger\dagger) \qquad v(x) = -2c_2 + c_2 e^{-x} = c_2(e^{-x} - 2).$$

Applying $(\dagger\dagger)$ and $(\dagger\dagger\dagger)$ we find

$$v'(\ell) + v(\ell) - 2v(0) = -c_2 e^{-\ell} + c_2(e^{-\ell} - 2) - 2c_2(-1) = 0,$$

and thus ③ is satisfied by $v(x) = c(e^{-x} - 2)$ for any choice of the constant c.

$$\boxed{\text{Conclusion: Solutions to the B.V.P. (*) are } \underline{\underline{\text{not}}} \text{ unique, for if } u = u(x) \text{ is a solution then so is } u = u(x) + c(e^{-x} - 2) \text{ for any constant } c.}$$

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#2 (cont.) (b) The general solution to the D.E. $u'' + u = f$ is $u = u_h + u_p$ where u_h is the general solution to the homogeneous equation $u'' + u = 0$ and u_p is a particular solution to $u'' + u = f$. Elementary O.D.E. techniques yield that $u_h = c_1 + c_2 e^{-x}$ and

$$u_p = \int_0^x f(t)dt - e^{-x}\int_0^x f(t)e^t dt. \text{ Thus}$$

$$(\dagger) \qquad u = u_h + u_p = c_1 + c_2 e^{-x} + \int_0^x f(t)dt - e^{-x}\int_0^x f(t)e^t dt$$

is the general solution to $u'' + u = f$, and

$$(\dagger\dagger) \qquad u' = -c_2 e^{-x} + f(x) - e^{-x} f(x) e^{x} + e^{-x}\int_0^x f(t)e^t dt$$

$$= -c_2 e^{-x} + e^{-x}\int_0^x f(t)e^t dt.$$

Substituting from $(\dagger)$ and $(\dagger\dagger)$ in the B.C. $u'(0) = u(0)$ yields $c_1 + c_2 = -c_2$, that is $c_1 = -2c_2$. Hence

$$(\dagger\dagger\dagger) \qquad u(x) = c_2(e^{-x} - 2) + \int_0^x f(t)dt - e^{-x}\int_0^x f(t)e^t dt.$$

Substituting from $(\dagger\dagger)$ and $(\dagger\dagger\dagger)$ into the B.C. $u(0) = \frac{1}{2}[u'(\ell) + u(\ell)]$ yields

$$-c_2 = \frac{1}{2}\left[\cancel{-c_2 e^{-\ell}} + \cancel{e^{-\ell}\int_0^\ell f(t)e^t dt} + \cancel{c_2 e^{-\ell}} - 2c_2 + \int_0^\ell f(t)dt - \cancel{e^{-\ell}\int_0^\ell f(t)e^t dt}\right]$$

or

$$-c_2 = -c_2 + \frac{1}{2}\int_0^\ell f(t)dt$$

or

$$0 = \int_0^\ell f(t)dt.$$

$$\boxed{\begin{array}{l}\underline{\text{Conclusion}}\text{: Solutions to the B.V.P. exist only if } \int_0^\ell f(t)dt = 0, \\ \text{and in that case } u(x) = c(e^{-x} - 2) + \int_0^x f(t)\left[1 - e^{t-x}\right]dt \text{ is} \\ \text{a solution for every constant } c.\end{array}}$$

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#3. Solve the B.V.P. $u''(x) = 0$ for $0 < x < 1$ with $u'(0) + ku(0) = 0$ and $u'(1) \pm ku(1) = 0$. Do the + and − cases separately. What is special about the case $k = 2$?

The general solution of the D.E. $u'' = 0$ on $(0,1)$ is $u(x) = c_1 x + c_2$ where c_1 and c_2 are arbitrary constants.

"+" $\boxed{\text{Case 1: } u'(0) + ku(0) = 0 \text{ and } u'(1) + ku(1) = 0}$.

In this case we have $0 = u'(0) + ku(0) = c_1 + kc_2$ and $0 = u'(1) + ku(1) = c_1 + k(c_1 + c_2)$. Substracting these two equations yields $c_1 k = 0$.

$\boxed{\text{If } k \neq 0}$ then $c_1 = 0 = c_2$ and $\boxed{u(x) \equiv 0 \text{ is the only solution}}$.

$\boxed{\text{If } k = 0}$ then $\boxed{u(x) \equiv \text{constant is a solution}}$ to the B.V.P. $u'' = 0$ on $0 < x < 1$ with $u'(0) = u'(1) = 0$.

unique solution

nonunique solutions

"−" $\boxed{\text{Case 2: } u'(0) + ku(0) = 0 \text{ and } u'(1) - ku(1) = 0}$.

In this case we have $0 = u'(0) + ku(0) = c_1 + kc_2$ and $0 = u'(1) - ku(1) = c_1 - k(c_1 + c_2)$. Adding these two equations gives $0 = (2 - k)c_1$.

unique solution

$\boxed{\text{If } k \neq 2, 0}$ then $c_1 = 0 = c_2$ and $\boxed{u(x) \equiv 0 \text{ is the only solution}}$

$\boxed{\text{If } k = 0}$ then $\boxed{u(x) \equiv \text{constant is a solution}}$ to the B.V.P. $u'' = 0$ on $0 < x < 1$ with $u'(0) = u'(1) = 0$.

nonunique solutions

$\boxed{\text{If } k = 2}$ then $0 = c_1 + 2c_2$ and $0 = -c_1 - 2c_2$. Thus $c_1 = -2c_2$ and c_2 is arbitrary. Therefore $\boxed{u = c_2(1 - 2x) \text{ is a solution}}$ to the B.V.P. $u'' = 0$ on $0 < x < 1$ with $u'(0) + 2u(0) = 0 = u'(1) - 2u(1)$; here $\boxed{c_2 \text{ is any constant}}$.

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#4. Consider the Neumann problem

$$\nabla^2 u = f(x,y,z) \quad \text{in } D,$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D.$$

(a) What can we surely add to any solution to get another solution? (So we don't have uniqueness.)

(b) Use the divergence theorem and the PDE to show that

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = 0$$

is a necessary condition for the Neumann problem to have a solution.

(c) Can you give a physical interpretation of part (a) and/or (b) for either heat flow or diffusion?

(a) If $u = u(x,y,z)$ is a solution to the Neumann problem then so is $u = u(x,y,z) + k$ for any constant k, since

$$\nabla^2 k = 0 \quad \text{and} \quad \frac{\partial k}{\partial n} = \nabla k \cdot n = 0.$$

(b) Suppose that the Neumann problem has a solution, say $u = u(x,y,z)$. Then

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_D \nabla^2 u\,dx\,dy\,dz$$

(because u is a solution to the PDE)

(cont.)

#4 (cont.)

$$= \iiint_D \nabla \cdot \nabla u \, dx\,dy\,dz$$

$$= \iint_{\partial D} \nabla u \cdot n \, dS \qquad \text{(by the divergence theorem)}$$

$$= \iint_{\partial D} \frac{\partial u}{\partial n} \, dS \qquad \left(\text{by definition: } \nabla u \cdot n = \frac{\partial u}{\partial n}\right)$$

$$= \iint_{\partial D} 0 \, dS \qquad \text{(by the boundary condition satisfied by } u\text{)}$$

$$= 0.$$

(c) The boundary condition $\underbrace{}_{}$ $\overbrace{\nabla u \cdot n = \frac{\partial u}{\partial n} = 0}$ corresponds to an insulated system in D if $u = u(x,y,z)$ represents temperature (i.e. heat flow is being modeled). Since the (heat) energy of the (isolated) system is conserved, the heat source/sink term $f(x,y,z)$ in the PDE should contribute no net heat energy change in the system, i.e. $\iiint_D f(x,y,z)\,dx\,dy\,dz$ should be zero.