Sec. 1.6, p. 31.

#1. What are the types of the following equations?

(a) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0.$

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0.$

(a) $A = 1,\ B = -4,\ C = 1$

$B^2 - 4AC = 16 - 4 = 12 > 0$ $\boxed{\text{hyperbolic}}$

(b) $A = 9,\ B = 6,\ C = 1$

$B^2 - 4AC = 36 - 36 = 0$ $\boxed{\text{parabolic}}$

#2. Find the regions in the xy-plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2 u_{yy} = 0$$

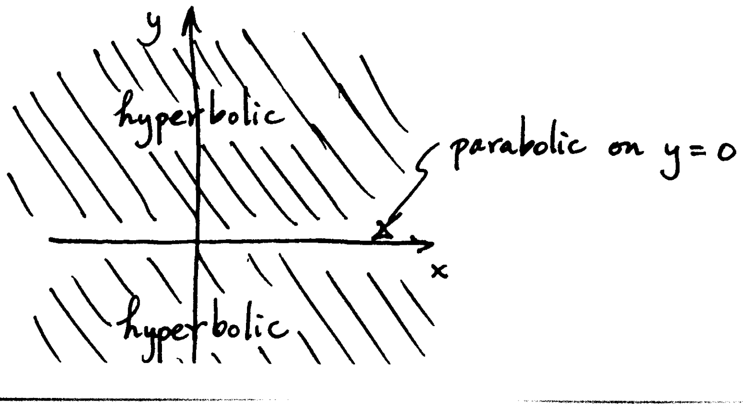
is elliptic, hyperbolic, or parabolic. Sketch them.

$A = 1+x,\ B = 2xy,\ C = -y^2.$

$$B^2 - 4AC = 4x^2y^2 + 4(1+x)y^2 = 4y^2(x^2+x+1) = 4y^2\left(x^2+x+\tfrac{1}{4}+\tfrac{3}{4}\right)$$

$$= 4y^2\left[\left(x+\tfrac{1}{2}\right)^2 + \tfrac{3}{4}\right] \geq 0$$ with equality if and only if $y = 0$.

$\boxed{\text{Therefore the equation is parabolic on the line } y=0 \text{ (the x-axis) and elsewhere in the xy-plane it is hyperbolic.}}$



Sec. 1.6, p. 31.

#4. What is the type of the equation

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0 \,?$$

Show by direct substitution that $u(x,y) = f(y+2x) + xg(y+2x)$ is a solution for arbitrary twice-differentiable functions f and g of a single variable.

$A = 1, \ B = -4, \ C = 4.$

$B^2 - 4AC = 16 - 16 = 0 \quad \boxed{\text{parabolic}}$

$u_x = 2f'(y+2x) + g(y+2x) + 2xg'(y+2x)$

$u_y = f'(y+2x) + xg'(y+2x)$

$u_{xy} = 2f''(y+2x) + g'(y+2x) + 2xg''(y+2x)$

$u_{xx} = 4f''(y+2x) + 4g'(y+2x) + 4xg''(y+2x)$

$u_{yy} = f''(y+2x) + xg''(y+2x)$

$$\therefore \ u_{xx} - 4u_{xy} + 4u_{yy} = (4-8+4)f''(y+2x) + (4-4)g'(y+2x) + (4-8+4)xg''(y+2x) \overset{\checkmark}{=} 0.$$

#6. Consider the equation $3u_y + u_{xy} = 0$.

(a) What is its type?

(b) Find the general solution.

(c) With the auxiliary conditions $u(x,0) = e^{-3x}$ and $u_y(x,0) = 0$, does a solution exist? Is it unique?

Sec. 1.6, p.31.

#6 (cont.) (a) $A = C = 0$, $B = 1$. $B^2 - 4AC = 1 > 0$ $\boxed{\text{hyperbolic}}$

(b) $3u_y + u_{xy} = 0$

$$(3u + u_x)_y = 0$$

$$3u + u_x = f(x)$$

(Linear first-order; integrating factor: $e^{\int 3\,dx} = e^{3x}$)

$$e^{3x}u_x + 3e^{3x}u = f(x)e^{3x}$$

$$\frac{\partial}{\partial x}\left(e^{3x}u\right) = f(x)e^{3x}$$

$$e^{3x}u = \int f(x)e^{3x}\,dx + g(y)$$

$$\boxed{u(x,y) = e^{-3x}\int f(x)e^{3x}\,dx + g(y)e^{-3x}}$$

(c) $e^{-3x} = u(x,0) = e^{-3x}\int f(x)e^{3x}\,dx + g(0)e^{-3x}$

$$1 = \int f(x)e^{3x}\,dx + g(0) \qquad (\text{for all real } x!)$$

Therefore $f = 0$ and $g(0) = 1$.

$$u_y(x,y) = g'(y)e^{-3x}$$

$$0 = u_y(x,0) = g'(0)e^{-3x}$$

Therefore $g'(0) = 0$.

<u>Conclusion</u>: Solutions to the I.V.P. exist. In fact, if g is any differentiable function of a single variable such that $g(0) = 1$ and $g'(0) = 0$, then $u(x,y) = g(y)e^{-3x}$ is a solution. Clearly solutions to the I.V.P. are not unique.

Supplementary problem for Sec 6.6.

Consider

$$(2) \qquad \left(\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}\right)^2 u = 0$$

in the plane $-\infty<x<\infty$, $-\infty<y<\infty$, where $\alpha^2+\beta^2>0$.

Let

$$\xi = \beta x-\alpha y,$$
$$\eta = \alpha x+\beta y.$$

Show that the general C^2-solution to (2) in the plane is

$$u = \eta f(\xi)+g(\xi)$$

where f and g are arbitrary C^2-functions of a single real variable.

We first solve for the variables x and y in terms of ξ and η. By Cramer's rule,

$$(*)\quad \left\{\begin{aligned} x &= \frac{\begin{vmatrix}\xi & -\alpha\\ \eta & \beta\end{vmatrix}}{\begin{vmatrix}\beta & -\alpha\\ \alpha & \beta\end{vmatrix}} = \frac{\beta\xi+\alpha\eta}{\alpha^2+\beta^2},\\ y &= \frac{\begin{vmatrix}\beta & \xi\\ \alpha & \eta\end{vmatrix}}{\begin{vmatrix}\beta & -\alpha\\ \alpha & \beta\end{vmatrix}} = \frac{\beta\eta-\alpha\xi}{\alpha^2+\beta^2}.\end{aligned}\right.$$

Next, we claim that as operators

$$(**) \qquad \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} = (\alpha^2+\beta^2)\frac{\partial}{\partial \eta}.$$

To see this, let $v = v(x,y)$ be any C^1-function in the plane. By the chain rule and $(*)$ we have

$$\frac{\partial v}{\partial \eta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \eta} = \frac{\alpha}{\alpha^2+\beta^2}\frac{\partial v}{\partial x} + \frac{\beta}{\alpha^2+\beta^2}\frac{\partial v}{\partial y},$$

and consequently,

$$(\alpha^2+\beta^2)\frac{\partial v}{\partial \eta} = \alpha\frac{\partial v}{\partial x} + \beta\frac{\partial v}{\partial y} = \left(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\right)v.$$

Since $v \in C^1(\mathbb{R}^2)$ was arbitrary, $(**)$ follows.

According to $(**)$, equation (2) is equivalent to

$$(\alpha^2+\beta^2)^2\frac{\partial^2 u}{\partial \eta^2} = 0.$$

Since $\alpha^2+\beta^2 > 0$, this means $u_{\eta\eta} = 0$. Integrating with respect to η once, holding ξ fixed, we find

$$u_\eta = c_1(\xi).$$

(The "constant" of integration may vary with ξ but is independent of η.)

Integrating again with respect to η, holding ξ fixed, yields

$$u = \eta c_1(\xi) + c_2(\xi)$$ (Again, the "constant" of integration may vary with ξ.)

In order for u to be a C^2-function in the plane, the functions c_1 and c_2 of the single variable ξ <u>must</u> be twice differentiable. Replacing c_1 and c_2 by the more appropriate function letters f and g, we have

$$\boxed{u = \eta f(\xi) + g(\xi)}$$

where f and g are C^2-functions on $\mathbb{R}$, or equivalently

$$\boxed{u(x,y) = (\alpha x + \beta y) f(\beta x - \alpha y) + g(\beta x - \alpha y)}.$$