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#1. Solve $u_{tt} = c^2 u_{xx}$ for $-\infty < x < \infty$, $-\infty < t < \infty$ such that $u(x,0) = e^x$ and $u_t(x,0) = \sin(x)$ for $-\infty < x < \infty$.

Solution: By d'Alembert's formula

$$u(x,t) = \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)\,d\xi$$

$$= \frac{1}{2}\left[e^{x+ct} + e^{x-ct}\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(\xi)\,d\xi$$

$$= e^x \cdot \left(\frac{e^{ct} + e^{-ct}}{2}\right) + \frac{1}{2c}\left[-\cos(x+ct) + \cos(x-ct)\right]$$

$$= e^x \cosh(ct) + \frac{1}{2c}\Big[-\cancel{\cos(x)\cos(ct)} + \sin(x)\sin(ct) + \cancel{\cos(x)\cos(ct)} + \sin(x)\sin(ct)\Big]$$

$$= \boxed{e^x \cosh(ct) + \frac{1}{c}\sin(x)\sin(ct)}$$

#3. The midpoint of a piano string of tension T, density ρ, and length l is hit by a hammer whose head diameter is $2a$. A flea is sitting at a distance $l/4$ from one end. (Assume that $a < l/4$; otherwise, poor flea!) How long does it take for the disturbance to reach the flea?

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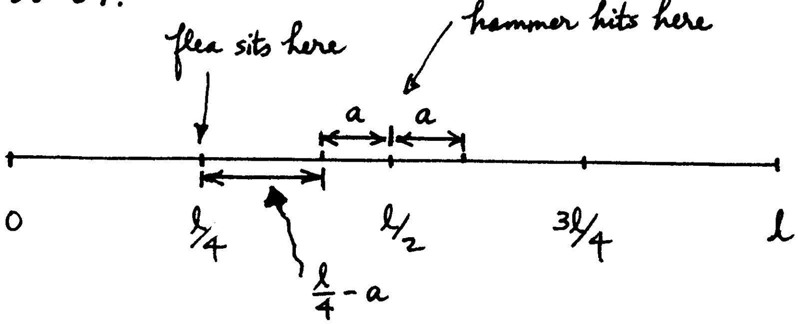
#3 (cont.)



The disturbance must travel the distance $\frac{\ell}{4} - a$ between the flea and the near edge of the hammer. The disturbance travels with speed $c = \sqrt{T/\rho}$. Therefore, the time it takes the flea to feel the disturbance is

$$\text{time} = \frac{\text{distance}}{\text{speed}} = \frac{\ell/4 - a}{c} = \boxed{\left(\frac{\ell}{4} - a\right)\sqrt{\rho/T}}.$$

#6. In exercise 5, find the greatest displacement,

$$\max_{-\infty < x < \infty} u(x,t),$$

as a function of t.

From exercise 5 we know that

$$u(x,t) = \frac{1}{2c} \text{ length of } \{ \overbrace{(x-ct, x+ct)}^{\text{interval of length } 2ct} \cap \underbrace{(-a,a)}_{\text{interval of length } 2a} \}$$

Therefore

$$\underset{-\infty < x < \infty}{\text{maximum}}\, u(x,t) = \frac{1}{2c} \text{ minumum of } \{2ct, 2a\}$$

$$= \frac{1}{2c} \begin{cases} 2ct & \text{if} \quad t \le a/c, \\ 2a & \text{if} \quad t > a/c, \end{cases}$$

$$= \begin{cases} t & \text{if} \quad t \le a/c, \\ a/c & \text{if} \quad t > a/c. \end{cases}$$

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#8. A spherical wave is a solution of the three-dimensional wave equation of the form $u = u(r,t)$, where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right) \qquad \text{(spherical wave equation).}$$

(a) Change variables $v = ru$ to get the equation for v: $v_{tt} = c^2 v_{rr}$.

(b) Solve for v using (3) and thereby solve the spherical wave equation.

(c) Use (8) to solve it with initial conditions $u(r,0) = \varphi(r)$, $u_t(r,0) = \psi(r)$, taking both φ and ψ to be even functions.

Solution:

(a) If $v = ru$ then $v_{tt} = ru_{tt}$, $v_r = (ru)_r = 1\cdot u + ru_r$,

and $v_{rr} = (u + ru_r)_r = u_r + ru_{rr} + 1\cdot u_r = 2u_r + ru_{rr}$.

Therefore $\boxed{v_{tt} =} ru_{tt} = rc^2\left(u_{rr} + \frac{2}{r}u_r\right) = c^2(ru_{rr} + 2u_r) = \boxed{c^2 v_{rr}}$.

(b) Thus by (3), the solution to $v_{tt} = c^2 v_{rr}$ is

$$v(r,t) = f(r+ct) + g(r-ct)$$

where f and g are arbitrary twice-differentiable function of a single variable. But $u = \frac{1}{r}v$ so

$$\boxed{u(r,t) = \frac{1}{r}f(r+ct) + \frac{1}{r}g(r-ct)}.$$

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#8 (c) (cont.) By d'Alembert's formula

$$v(r,t) = \frac{1}{2}\left[\Phi(r+ct) + \Phi(r-ct)\right] + \frac{1}{2c}\int_{r-ct}^{r+ct} \Psi(\xi)\,d\xi$$

where $v(r,0) = \Phi(r) = r\,u(r,0) = r\varphi(r)$ and $v_t(r,0) = \Psi(r) = r\,u_t(r,0) = r\psi(r)$. Thus

$$v(r,t) = \frac{1}{2}\left[(r+ct)\varphi(r+ct) + (r-ct)\varphi(r-ct)\right] + \frac{1}{2c}\int_{r-ct}^{r+ct} \xi\psi(\xi)\,d\xi$$

and, consequently,

$$u(r,t) = \frac{1}{r}v(r,t)$$

$$= \boxed{\frac{1}{2r}\left[(r+ct)\varphi(r+ct) + (r-ct)\varphi(r-ct)\right] + \frac{1}{2cr}\int_{r-ct}^{r+ct} \xi\psi(\xi)\,d\xi .}$$

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#9. Solve $\begin{cases} u_{xx} - 3u_{xt} - 4u_{tt} = 0 & \text{for } -\infty < x < \infty,\ t > 0 \\ u(x,0) = x^2 \text{ and } u_t(x,0) = e^x & \text{for } -\infty < x < \infty. \end{cases}$

$$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x \partial t} - 4\frac{\partial^2 u}{\partial t^2} = 0$$

$$\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)\overbrace{\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)u}^{v} = 0$$

Let $v = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}$ in the above P.D.E. Then we must solve $\frac{\partial v}{\partial x} - 4\frac{\partial v}{\partial t} = 0$. By standard techniques,

$$v(x,t) = f(t+4x)$$

where f is an arbitrary differentiable function of a single variable. But $v = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}$ so we must solve

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = f(t+4x).$$

By standard techniques

$$u(x,t) = F(t+4x) + G(t-x)$$

where F and G are arbitrary twice-differentiable functions of a single variable. We must determine F and G so that $u = u(x,t)$ satisfies the I.C.'s $u(x,0) = x^2$ and $u_t(x,0) = e^x$ for $-\infty < x < \infty$:

① $x^2 = u(x,0) = F(4x) + G(-x)$,

② $e^x = u_t(x,0) = F'(4x) + G'(-x)$.

#9 (cont.) Integrating ② yields

$$\int_0^x e^{\xi}\,d\xi = \int_0^x [F'(4\xi) + G'(-\xi)]\,d\xi = \frac{1}{4}\int_0^x F'(4\xi)\,d\xi - \int_0^x G'(\xi)(-d\xi)$$

$$e^x - 1 = \frac{1}{4}\int_0^{4x} F'(w)\,dw - \int_0^{-x} G'(w)\,dw = \frac{1}{4}[F(4x) - F(0)] - [G(-x) - G(0)]$$

$$\left.\begin{array}{lrl} ②' & 4(e^x - 1) + F(0) - 4G(0) = & F(4x) - 4G(-x) \\ ① & x^2 = & F(4x) + G(-x) \end{array}\right\}.$$

Subtracting ① from ②' yields

$$4(e^x - 1) + F(0) - 4G(0) - x^2 = -5G(-x)$$

or equivalently

$$③ \qquad G(s) = -\frac{4}{5}\left(e^{-s} - 1\right) - \frac{1}{5}F(0) + \frac{4}{5}G(0) + \frac{1}{5}s^2.$$

Also, multiplying ① by 4 and adding to ②' gives

$$4(e^x - 1) + F(0) - 4G(0) + 4x^2 = 5F(4x)$$

or equivalently

$$④ \qquad F(s) = \frac{4}{5}\left(e^{s/4} - 1\right) + \frac{1}{5}F(0) - \frac{4}{5}G(0) + \frac{s^2}{20}.$$

Substituting from ③ and ④ we have

$$u(x,t) = F(t + 4x) + G(t - x)$$

$$= \frac{4}{5}\left(e^{x + \frac{t}{4}} \cancel{-1}\right) + \cancel{\frac{1}{5}F(0)} - \cancel{\frac{4}{5}G(0)} + \frac{(t+4x)^2}{20}$$

$$+ \left(-\frac{4}{5}\right)\left(e^{x-t} \cancel{-1}\right) - \cancel{\frac{1}{5}F(0)} + \cancel{\frac{4}{5}G(0)} + \frac{1}{5}(t-x)^2.$$

#9 (cont.) $u(x,t) = \frac{4}{5}e^{x}\left(e^{\frac{t}{4}} - e^{-t}\right) + \frac{1}{20}\left[(t+4x)^2 + 4(t-x)^2\right]$

$$u(x,t) = \frac{4}{5}e^{x}\left(e^{\frac{t}{4}} - e^{-t}\right) + \frac{1}{20}\left[t^2 + \cancel{8xt} + 16x^2 + 4t^2 - \cancel{8xt} + 4x^2\right]$$

$$u(x,t) = \frac{4}{5}e^{x}\left(e^{\frac{t}{4}} - e^{-t}\right) + \frac{1}{20}\left[5t^2 + 20x^2\right]$$

$$\boxed{u(x,t) = \frac{4}{5}e^{x}\left(e^{\frac{t}{4}} - e^{-t}\right) + x^2 + \frac{1}{4}t^2}$$