Sec. 2.2, p. 40.

#2. For a solution $u(x,t)$ of the wave equation with $\rho = T = c = 1$, the energy density is defined as $e = \frac{1}{2}(u_t^2 + u_x^2)$ and the momentum density as $p = u_t u_x$.

(a) Show that $\frac{\partial e}{\partial t} = \frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial t} = \frac{\partial e}{\partial x}$.

(b) Show that both $e = e(x,t)$ and $p = p(x,t)$ also satisfy the wave equation.

(a) Since $u = u(x,t)$ is a solution of the wave equation with $c=1$,

$$(*) \qquad u_{tt} - u_{xx} = 0.$$

We compute as follows.

$$(+) \quad \frac{\partial e}{\partial t} = \frac{\partial}{\partial t}\left[\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2\right] = u_t u_{tt} + u_x u_{xt}$$

Same!

$$(++) \quad \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left[u_t u_x\right] = u_{tx}u_x + u_t u_{xx} \underset{\text{by }(*)}{=} u_{xt}u_x + u_t u_{tt}$$

$$(+++) \quad \frac{\partial p}{\partial t} = \frac{\partial}{\partial t}\left[u_t u_x\right] = u_{tt}u_x + u_t u_{xt}$$

Same!

$$(++++) \quad \frac{\partial e}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2\right] = u_t u_{tx} + u_x u_{xx} \underset{\text{by }(*)}{=} u_t u_{xt} + u_x u_{tt}$$

(b) In this part of the problem we assume that $u = u(x,t)$ is thrice continuously differentiable. In particular, third order partial derivatives of u exist (e.g. u_{ttt}, u_{xxx}, u_{ttx}, ...) and mixed partials (like u_{ttx} and u_{xtt}) are equal. It follows that $p_{tx} = p_{xt}$ and $e_{tx} = e_{xt}$.

We compute as follows:

#2 (cont.)

$$e_{tt} - e_{xx} = (e_t)_t - (e_x)_x \overset{\text{From (a)}}{=} (p_x)_t - (p_t)_x = p_{xt} - p_{tx} = 0$$

$$p_{tt} - p_{xx} = (p_t)_t - (p_x)_x \underset{\text{From (a)}}{=} (e_x)_t - (e_t)_x = e_{xt} - e_{tx} = 0$$

Therefore $e = e(x,t) = \frac{1}{2}(u_t^2 + u_x^2)$ and $p = p(x,t) = u_t u_x$ are solutions to the wave equation with $c = 1$.

#3. Show that the wave equation (solutions) have the following invariance properties. (Let $u = u(x,t)$ be any solution to the wave equation $u_{tt} - c^2 u_{xx} = 0$.)

(a) Any translate $v = u(x-y, t)$, where y is fixed, is also a solution (to the wave equation).

(b) Any derivative, say u_x, of a solution is also a solution.

(c) The dilated function $v = u(ax, at)$ is also a solution.

(a) Let $v(x,t) = u(x-y, t)$ where y is a fixed number. Then

$$v_{tt}(x,t) - c^2 v_{xx}(x,t) = u_{tt}(x-y,t) - c^2 u_{xx}(x-y,t) = 0$$

because u is a solution to the wave equation at every point in the xt-plane (in particular, at $(x-y,t)$).

(b) Let $v(x,t) = u_x(x,t)$ where $u = u(x,t)$ is a thrice continuously differentiable function of two variables that

#3 (cont.) is a solution of the wave equation. Then

$$v_{tt} - c^2 v_{xx} = u_{xtt} - c^2 u_{xxx} = (\underbrace{u_{tt} - c^2 u_{xx}}_{0})_x = 0$$

and similarly for $v(x,t) = u_t(x,t)$.

(c) Let $v(x,t) = u(ax, at)$. Then by the chain rule we have:

$$v_t(x,t) = a u_t(ax, at),$$

$$v_{tt}(x,t) = a^2 u_{tt}(ax, at),$$

$$v_x(x,t) = a u_x(ax, at),$$

$$v_{xx}(x,t) = a^2 u_{xx}(ax, at).$$

Consequently

$$\begin{aligned} v_{tt}(x,t) - c^2 v_{xx}(x,t) &= a^2 u_{tt}(ax,at) - c^2 a^2 u_{xx}(ax,at) \\ &= a^2 \big(\overbrace{u_{tt}(ax,at) - c^2 u_{xx}(ax,at)}^{0} \big) \\ &= 0. \end{aligned}$$

Sec. 2.2, p. 40.

#5. For the damped string, $\rho u_{tt} - T u_{xx} + R u_t = 0$ where R is a positive constant, show that the energy at time t

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2)\,dx$$

is a decreasing function.

We show that $E'(t) \le 0$ for all $t \ge 0$, and hence E is a decreasing (nonincreasing) function of time.

$$E'(t) = \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2)\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}(\rho u_t^2 + T u_x^2)\,dx$$

$$= \int_{-\infty}^{\infty} (\rho u_t u_{tt} + T u_x u_{tx})\,dx$$

$$= \int_{-\infty}^{\infty} [u_t(T u_{xx} - R u_t) + T u_x u_{tx}]\,dx \qquad \text{(from damped wave equation)}$$

$$= \int_{-\infty}^{\infty} T u_t u_{xx}\,dx - \int_{-\infty}^{\infty} R u_t^2\,dx + T\int_{-\infty}^{\infty} u_x u_{tx}\,dx$$

Integrating the last term by parts with $U = u_x$ and $dV = u_{tx}\,dx$, we have

$$E'(t) = \cancel{\int_{-\infty}^{\infty} T u_t u_{xx}\,dx} - \int_{-\infty}^{\infty} R u_t^2\,dx + T\Big[\lim_{x\to\infty} \overbrace{u_x(x,t)}^{\text{bounded}}\overbrace{u_t(x,t)}^{\text{goes to zero}} -$$

$$\lim_{x\to-\infty} \overbrace{u_x(x,t)}^{\text{bounded}}\overbrace{u_t(x,t)}^{\text{goes to zero}}\Big] - T\cancel{\int_{-\infty}^{\infty} u_t u_{xx}\,dx} \qquad \text{(cont.)}$$

Therefore $E'(t) = -R\int_{-\infty}^{\infty} u_t^2\,dx \le 0$.

#6 Prove that, among all possible dimensions, only in three dimensions can one have distortionless spherical wave propagation with attenuation. This means the following. A spherical wave in n-dimensional space satisfies the PDE

$$u_{tt} - c^2\left(u_{rr} + \frac{n-1}{r}u_r\right) = 0$$

where r is the spherical coordinate. Consider such a wave that has the special form $u(r,t) = \alpha(r)f(t-\beta(r))$, where $\alpha(r)$ is the attenuation and $\beta(r)$ the delay. The question is whether such solutions exist for "arbitrary" functions f.

(a) Plug the special form into the PDE to get an ODE for f.
(b) Set the coefficients of f'', of f', and of f equal to zero.
(c) Solve the ODEs to see that $n=1$ or $n=3$ (unless $u \equiv 0$).
(d) If $n=1$, show that $\alpha(r)$ is a constant (so that "there is no attenuation").

(a) $u_t = \alpha(r)f'(t-\beta(r))$, $\quad u_{tt} = \alpha(r)f''(t-\beta(r))$

$$u_r = \alpha'(r)f(t-\beta(r)) + \alpha(r)f'(t-\beta(r))(-\beta'(r))$$

$$u_{rr} = \alpha''(r)f(t-\beta(r)) + 2\alpha'(r)f'(t-\beta(r))(-\beta'(r)) + \alpha(r)\left[f'(t-\beta(r))(-\beta''(r)) + f''(t-\beta(r))(\beta'(r))^2\right]$$

#6 (cont.) $\quad 0 = u_{tt} - c^2\left(u_{rr} + \frac{n-1}{r} u_r\right)$

$$0 = \alpha(r) f''(t-\beta(r)) - c^2\Big\{ \alpha''(r) f(t-\beta(r)) - (2\alpha'(r)\beta'(r) + \alpha(r)\beta''(r)) f'(t-\beta(r))$$

$$+ \alpha(r)(\beta'(r))^2 f''(t-\beta(r)) + \frac{n-1}{r}\big(\alpha'(r) f(t-\beta(r)) - \alpha(r)\beta'(r) f'(t-\beta(r))\big)\Big\}$$

$$0 = \left[\alpha(r) - c^2\alpha(r)(\beta'(r))^2\right] f''(t-\beta(r))$$

$$+ \left[c^2\left(2\alpha'(r)\beta'(r) + \alpha(r)\beta''(r) + \frac{(n-1)}{r}\alpha(r)\beta'(r) \right)\right] f'(t-\beta(r))$$

$$+ \left[-c^2\left(\alpha''(r) + \frac{n-1}{r}\alpha'(r)\right)\right] f(t-\beta(r))$$

(b) Setting the coefficients of f'', f', and f equal to zero (because f is "arbitrary") yields the coupled system of three ODEs:

$$\begin{cases} \alpha(r)\left[1 - c^2(\beta'(r))^2\right] = 0 & (*) \\ 2\alpha'(r)\beta'(r) + \alpha(r)\beta''(r) + \frac{(n-1)}{r}\alpha(r)\beta'(r) = 0 & (**) \\ \alpha''(r) + \frac{(n-1)}{r}\alpha'(r) = 0 & (***) \end{cases}$$

(c) In the third $\overbrace{\text{equation}}^{(***)}$, let $u(r) = \alpha'(r)$; then multiply by the integrating factor r^{n-1} to obtain

$$r^{n-1} u'(r) + (n-1) r^{n-2} u(r) = 0$$

$$\Rightarrow \quad \frac{d}{dr}\left[r^{n-1} u(r) \right] = 0$$

#6(c) (cont.) Therefore $[\alpha'(r) =]\ u(r) = c_1 r^{1-n}$ $\quad(****)$

Case $n=2$: $\quad \alpha'(r) = c_1 r^{-1} \Rightarrow \alpha(r) = c_1 \log(r) + c_2$. $\quad(*****)$

Case $n \neq 2$: $\quad \alpha'(r) = c_1 r^{1-n}\ (1-n \neq -1) \Rightarrow \alpha(r) = \dfrac{c_1 r^{2-n}}{2-n} + c_2$. $\left(\begin{smallmatrix}***\\***\end{smallmatrix}\right)$

Therefore the function α is either the zero function ($c_1 = c_2 = 0$) or has at most one real zero. From equation (*) on the previous page, it follows $\beta'(r) = \pm\frac{1}{c}$ for all (except possibly one) $r > 0$. Hence $\beta''(r) = 0$ for such $r > 0$, so by equation (**)

$$2\alpha'(r) + \frac{n-1}{r}\alpha(r) = 0 \quad \text{for all (save possibly one) } r > 0. \left(\begin{smallmatrix}****\\***\end{smallmatrix}\right)$$

Substituting from (****) and either (*****) or $\left(\begin{smallmatrix}***\\***\end{smallmatrix}\right)$ into $\left(\begin{smallmatrix}****\\***\end{smallmatrix}\right)$, and multiplying through by r yields the following.

Case $n = 2$: $\quad 2c_1 + c_1\log(r) + c_2 = 0$ for all (save possibly one) $r > 0$.

Case $n \neq 2$: $\quad 2c_1 r^{2-n} + \left(\dfrac{n-1}{2-n}\right)c_1 r^{2-n} + (n-1)c_2 = 0$ " " " ".

Clearly $c_1 = c_2 = 0$ in the case $n=2$; i.e. $\boxed{\alpha(r) \equiv 0 \text{ when } n = 2}$.

In the case $n \neq 2$, we must have $\left(2 + \dfrac{n-1}{2-n}\right)c_1 = (n-1)c_2 = 0$. $\left(\begin{smallmatrix}****\\****\end{smallmatrix}\right)$

If $n=1$ then $\left(\begin{smallmatrix}****\\****\end{smallmatrix}\right)$ implies $c_1 = 0$ and c_2 is arbitrary; i.e. $\boxed{\alpha(r) \equiv c_2 = \text{constant}}$.

If $n=3$ then " " $c_2 = 0$ " c_1 " "; i.e. $\boxed{\alpha(r) \equiv -c_1 r^{-1}}$.

If $n \neq 1,2,3$ then " " $c_1 = c_2 = 0$; i.e. $\boxed{\alpha(r) \equiv 0}$.