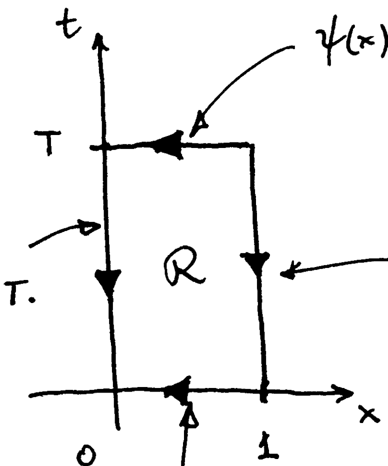
Sec. 2.3, pp. 44–45.

\#1. Consider the solution $u(x,t) = 1 - x^2 - 2kt$ of the diffusion equation $u_t - k u_{xx} = 0$. Find the locations of its maximum and minimum in the closed rectangle $\overline{R} = \{(x,t) : 0 \le x \le 1,\ 0 \le t \le T\}$.

Since $u_t = -2k \neq 0$, the maximum and minimum of u on $\overline{R}$ cannot occur at an interior point of $\overline{R}$. The values of u on $\partial\overline{R}$ behave as follows (arrows show the directions of increasing u).

$\psi(x) = u(x,T) = 1 - x^2 - 2kT$ is decreasing on $0 \le x \le 1$.

$h(t) = u(0,t) = 1 - 2kt$ is decreasing on $0 \le t \le T$.



$g(t) = u(1,t) = -2kt$ is decreasing on $0 \le t \le T$.

$\varphi(x) = u(x,0) = 1 - x^2$ is decreasing on $0 \le x \le 1$.

Therefore
$$\max_{(x,t)\in\overline{R}} u(x,t) = u(0,0) = 1$$
and
$$\min_{(x,t)\in\overline{R}} u(x,t) = u(1,T) = -2kT.$$

(Note that the maximum and minimum values of u occur on the initial and side walls of $\overline{R}$, in accordance with the maximum/minimum principle.)

Sec. 2.3, pp. 44-45.

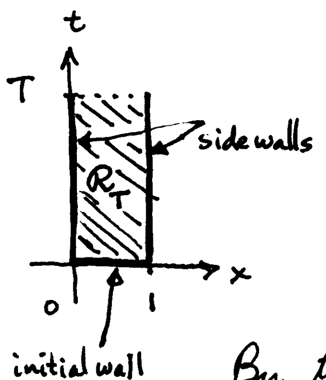
#3. Consider the diffusion equation $u_t - u_{xx} = 0$ for $0<x<1$ and $0<t<\infty$, with $u(0,t) = u(1,t) = 0$ for $t \geq 0$ and $u(x,0) = 1-x^2$ for $0 \leq x \leq 1$. Note that this initial function does not satisfy the boundary condition at the left end, but that the solution will satisfy it for all $t>0$.

(a) Show that $u(x,t) > 0$ at all interior points $0<x<1$, $0<t<\infty$.

(b) For each $t>0$, let $\mu(t)$ denote the maximum of $u(x,t)$ for $0 \leq x \leq 1$. Show that μ is a nonincreasing function of t.

(c) Draw a rough sketch of what you think the solution looks like at a few times.

(a) Fix $T>0$. Let



$$R_T = \{(x,t) : 0<x<1,\ 0<t<T\}$$

m_T = minimum of $u(x,t)$ on the sidewalls and initial wall of R_T,

M_T = maximum of $u(x,t)$ " " " " " " " ".

By the strong maximum/minimum principle

$$m_T \leq u(x,t) \leq M_T$$

for all $(x,t) \in R_T$, with equality only if u is a constant function on $\overline{R_T}$. However $u(x,0) = 1-x^2 \not\equiv$ constant for $0 \leq x \leq 1$. Therefore $m_T < u(x,t) < M_T$ for all $(x,t) \in R_T$. It is clear that $m_T = 0$ and $M_T = 1$, and hence

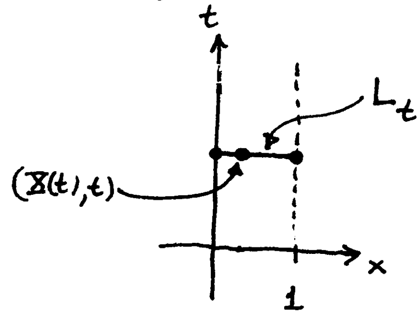
$$(*) \qquad 0 < u(x,t) < 1$$

for all $(x,t) \in R_T$. Because $T>0$ is arbitrary, $(*)$ holds for all

Sec. 2.3, pp. 44–45.

#3 (cont.) points (x,t) such that $0 < x < 1$ and $0 < t < \infty$.

(b) Let $\Xi(t)$ denote the number in $[0,1]$ such that $\mu(t) = u(\Xi(t), t)$, i.e. the maximum of $u(x,t)$ on the horizontal line segment $L_t = \{(x,t) : 0 \le x \le 1\}$ occurs at the point $(\Xi(t), t)$.



(If the maximum of $u(x,t)$ on L_t occurs at more than one point, then take $(\Xi(t), t)$ as the left-most point on L_t where the maximum of $u(x,t)$ occurs.)

By the chain rule applied to $\mu(t) = u(\Xi(t), t)$, we have

$$\frac{d\mu}{dt} = \frac{\partial u}{\partial \Xi}\cdot\frac{d\Xi}{dt} + \frac{\partial u}{\partial t}\cdot\overbrace{\frac{dt}{dt}}^{1}$$

$$= \frac{\partial u}{\partial \Xi}\cdot\frac{d\Xi}{dt} + \frac{\partial^2 u}{\partial x^2}$$

since u is a solution to $u_t - u_{xx} = 0$. But $\frac{\partial u}{\partial x}(\Xi(t), t) = 0$ and $\frac{\partial^2 u}{\partial x^2}(\Xi(t), t) \le 0$ since the maximum of u on L_t occurs at $(\Xi(t), t)$. Consequently

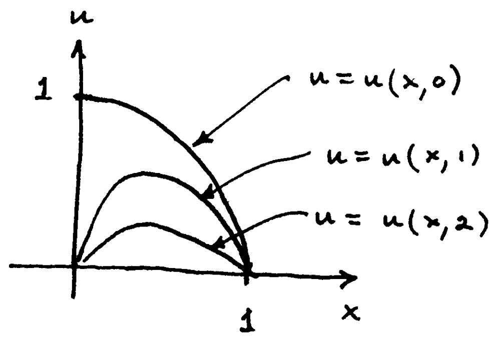
$$\frac{d\mu}{dt} = 0\cdot\frac{d\Xi}{dt} + \frac{\partial^2 u}{\partial x^2} \le 0\,;$$

i.e. μ is a nonincreasing function of t.

Sec 2.3, pp. 44-45.

3 (cont.)

(c)



#5. The purpose of this exercise is to show that the conclusion of the maximum principle need not hold for the equation $u_t = x u_{xx}$, which has a variable coefficient.

(a) Verify that $u(x,t) = -2xt - x^2$ is a solution. Find the location of its maximum in the rectangle $\overline{R} = \{(x,t) : -2 \le x \le 2,\ 0 \le t \le 1\}$.

(b) Where precisely does our proof of the maximum principle break down for this equation?

(a) If $u(x,t) = -2xt - x^2$ then

$$u_t - x u_{xx} = -2x - x(-2) \overset{\checkmark}{=} 0.$$

Note that $u_x = -2t - 2x$ and $u_t = -2x$. Therefore if the maximum of u occurs at an interior point of $\overline{R}$, then $u_t = u_x = 0$ there. But $u_x = u_t = 0$ implies $x = t = 0$, and $(0,0)$ is not an interior point of $\overline{R}$. Therefore the maximum of u on $\overline{R}$ must occur at a boundary point.

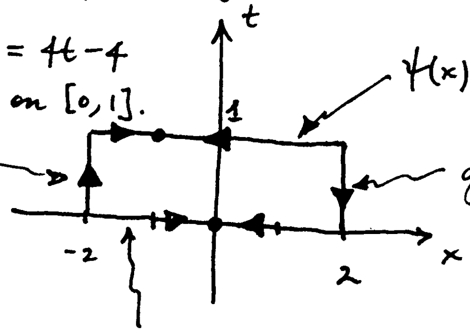
On the boundary of $\overline{R}$, the function $u(x,t)$ has the

Sec. 2.3, pp. 44-45.

#5 (cont.) following behaviors (arrows indicate the direction of increasing u).

$h(t) = u(-2,t) = 4t-4$ is increasing on $[0,1]$.

$\psi(x) = u(x,1) = -2x - x^2$ is increasing on $[-2,-1]$ and decreasing on $[-1,2]$.



$g(t) = u(2,t) = -4t-4$ is decreasing on $[0,1]$.

$\varphi(x) = u(x,0) = -x^2$ is increasing on $[-2,0]$ and decreasing on $[0,2]$.

$u(0,0) = 0 - 0 = 0 \quad , \quad u(-1,1) = 2 - 1 = 1$

Thus the maximum of u on the rectangle occurs at $(-1,1)$, a point on the back wall $t=1$. Thus the maximum of u on $\overline{R}$ does not occur at a point on the initial or side walls of $\overline{R}$, in opposition to the conclusion of the maximum principle.

(b) The analogue of the diffusion inequality (2), p. 42, does not hold for this equation on the rectangle R:

$$v_t - x v_{xx} = (u+\varepsilon x^2)_t - x(u+\varepsilon x^2)_{xx} = u_t - x(u_{xx} + 2\varepsilon)$$

$$= \underbrace{u_t - x u_{xx}}_{0} - 2x\varepsilon$$

$$= \underbrace{-2x\varepsilon}_{\text{this need not be negative for all } -2 \le x \le 2.}$$

Sec. 2.3, pp. 44-45.

#8. Consider the diffusion equation $u_t - ku_{xx} = 0$ on $0 < x < \ell$ with the Robin boundary conditions $u_x(0,t) - a_0 u(0,t) = 0$ and $u_x(\ell,t) + a_\ell u(\ell,t) = 0$. If $a_0 > 0$ and $a_\ell > 0$, use the energy method to show that the endpoints contribute to the decrease of $\int_0^\ell u^2(x,t)\,dx$. (This is interpreted to mean that part of the "energy" is lost at the boundary, so we call the boundary conditions "radiating" or "dissipative".)

Let $E(t) = \int_0^\ell u^2(x,t)\,dx$. Then

$$\frac{dE}{dt} = \frac{d}{dt}\int_0^\ell u^2(x,t)\,dx$$

$$= \int_0^\ell \frac{\partial}{\partial t}\left\{u^2(x,t)\right\}dx$$

$$= \int_0^\ell 2u(x,t)\,u_t(x,t)\,dx$$

$$= \int_0^\ell \overbrace{2u(x,t)k}^{v}\,\overbrace{u_{xx}(x,t)\,dx}^{dV} \qquad (\text{Because } u_t - ku_{xx} = 0.)$$

$$= \overbrace{2ku(x,t)}^{v}\,\overbrace{u_x(x,t)}^{V}\Bigg|_{x=0}^{x=\ell} - \int_0^\ell \overbrace{2k\,u_x(x,t)}^{v'}\,\overbrace{u_x(x,t)}^{V}\,dx$$

$$= 2k\Big(u(\ell,t)u_x(\ell,t) - u(0,t)u_x(0,t)\Big) - 2k\int_0^\ell u_x^2(x,t)\,dx$$

$$= \underbrace{2k\Big(-a_\ell u^2(\ell,t) - a_0 u^2(0,t)\Big)}_{\substack{\text{the contribution of the endpoints}\\ \text{to the loss in } E(t).}} - 2k\int_0^\ell u_x^2(x,t)\,dx$$