#1. Solve the diffusion equation $u_t - ku_{xx} = 0$ for $-\infty < x < \infty, 0 < t < \infty$, with the initial condition $u(x,0) = q(x)$ for $-\infty < x < \infty$ where

$$q(x) = \begin{cases} 
1 & \text{if } |x| < l, \\
0 & \text{if } |x| > l.
\end{cases}$$

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} q(y) dy$$

$$= \int_{-l}^{l} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} dy$$

--- let $p = \frac{x-y}{\sqrt{4kt}}$. then $dp = \frac{-dy}{\sqrt{4kt}}$

$$= \frac{1}{\sqrt{\pi}} \int_{-\frac{x-l}{\sqrt{4kt}}}^{\frac{x+l}{\sqrt{4kt}}} e^{-p^2} dp$$

$$= \frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \left[ \int_{0}^{\frac{x+l}{\sqrt{4kt}}} - \int_{0}^{\frac{x-l}{\sqrt{4kt}}} \right] e^{-p^2} dp$$

$$= \frac{1}{2} \left( \text{Erf} \left( \frac{x+l}{\sqrt{4kt}} \right) - \text{Erf} \left( \frac{x-l}{\sqrt{4kt}} \right) \right)$$
Sec. 2.4, p. 50-52.

#3 Solve \( u_t - ku_{xx} = 0 \) for \( -\infty < x < \infty, 0 < t < \infty \),
given that \( u(x,0) = \varphi(x) = e^{3x} \) for \( -\infty < x < \infty \).

By (8), p. 418 the solution is

\[
u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y) dy
\]

\[
= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{x^2 - 2xy + y^2}{4kt}} e^{3y} dy
\]

\[
= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{y^2 - 2y(x+6kt) + (x+6kt)^2}{4kt} - 12xkt - 36k^2t^2}{\frac{+12xkt + 36k^2t^2}{4kt}} e^{-\frac{(y-(x+6kt))^2}{4kt}} dy
\]

\[
= e^{\frac{3x+9kt}{\sqrt{\pi}}} \int_{-\infty}^{\infty} e^{-u^2} du
\]

\[
= e^{\frac{3x+9kt}{\sqrt{\pi}}} \quad (by \ exercise \ #7, \ p. 51)
\]
Sec. 2.4, pp. 50–52.

#5. Prove properties (a) to (e) of (solution) to the diffusion equation
\[ u_t - ku_{xx} = 0, \quad -\infty < x < \infty, \quad 0 < t < \infty. \]

(a) The translate \( v(x,t) = u(x-y,t) \) of any solution \( u = u(x,t) \) is another solution, for any fixed \( y \).

(b) Any derivative, \( v(x,t) = u_x(x,t) \) or \( w(x,t) = u_t(x,t), \) of a solution \( u = u(x,t) \) is again a solution. (Of course, we assume that the solution \( u \) is sufficiently differentiable.)

(c) A linear combination \( w(x,t) = c_1 u(x,t) + c_2 v(x,t) \) of solutions \( u = u(x,t) \) and \( v = v(x,t) \) is again a solution.

(d) If \( S = S(x,t) \) is a solution and \( g = g(y) \) is "any" function of a single variable, then
\[ v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)dy \]
is a solution, as long as this improper integral converges "appropriately".

(e) If \( u = u(x,t) \) is a solution and \( a > 0 \) is a constant then the dilated function \( v = u(ax,at) \) is a solution.

(a) \( v_t - kv_{xx} = u_t(x-y,t) - ku_{xx}(x-y,t) = 0. \)

(b) \( v_t - kv_{xx} = u_t - ku_{xx} = (u_t - ku_{xx}) = 0. \)

\( w_t - kw_{xx} = u_{tt} - ku_{txx} = (u_{tt} - ku_{txx}) = 0. \)

(c) \( w_t - kw_{xx} = (c_1 u_x + c_2 v_x) - k(c_1 u_x + c_2 v_x) = c_1 (u_t - ku_{xx}) + c_2 (v_t - kv_{xx}) = 0. \)

(d) \( v_t - kv_{xx} = \int_{-\infty}^{\infty} S_t(x-y,t)g(y)dy - k\int_{-\infty}^{\infty} S_{xx}(x-y,t)g(y)dy \)
\[ = \left[ \int_{-\infty}^{\infty} S(x-y,t)g(y)dy \right] - k\int_{-\infty}^{\infty} S(x-y,t)g(y)dy = 0. \)
#5 (cont.) \( v(x,t) = u(\sqrt{a}x, at) \)

\[
\begin{align*}
\frac{\partial v}{\partial x} &= \frac{\partial u}{\partial x} \frac{\sqrt{a}}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\sqrt{a}}{\partial \eta} = \sqrt{a} u_x(\sqrt{a}x, at) \\
\frac{\partial^2 v}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial u_x}{\partial x} \frac{\partial \sqrt{a}}{\partial x} + \frac{\partial u_x}{\partial \eta} \frac{\partial \sqrt{a}}{\partial \eta} \right) = au_{xx}(\sqrt{a}x, at) \\
\frac{\partial v}{\partial t} &= \frac{\partial u}{\partial x} \frac{\partial \sqrt{a}}{\partial x} \frac{\partial t}{\partial t} + \frac{\partial u}{\partial \eta} \frac{\partial \sqrt{a}}{\partial \eta} \frac{\partial t}{\partial t} = au_t(\sqrt{a}x, at)
\end{align*}
\]

Therefore \( v_t(x,t) - kv_{xx}(x,t) = au_t(\sqrt{a}x, at) - au_{xx}(\sqrt{a}x, at) \)

\[
= a(u_t - ku_{xx})(\sqrt{a}x, at)
\]

\[= 0. \]

---

#9. Solve the diffusion equation \( u_t - ku_{xx} = 0 \) for \(-\infty < x < \infty, \ 0 < t < \infty, \) with the initial condition \( u(x,0) = x^2 \) for \(-\infty < x < \infty.\)

---

Let \( v(x,t) = u_{xx}(x,t). \) Then

\[
\begin{align*}
v_t - kv_{xx} &= u_{xxxx} - ku_{xxxx} = (u_t - ku_x)_{xxxx} = (0)_{xxxx} = 0
\end{align*}
\]

and

\[
v(x,0) = u_{xxx}(x,0) = \frac{d^3}{dx^3}(x^2) = 0.
\]

By a uniqueness theorem (for example, see F. John, Partial Differential Equations (4th ed.), pp. 216-218), it follows that the only solution of this initial value problem which obeys \( v(x,t) \leq M e^{-ax^2} \)

for all \(-\infty < x < \infty, \ 0 < t < \infty, \) is the trivial solution \( v(x,t) \equiv 0. \)

Integrating the zero solution three times with respect to \( x \) (holding \( t \) fixed), we find that

\[
u(x,t) = A(t)x^2 + B(t)x + C(t)
\]
sec. 2.4, pp. 50-52.

#9 (cont.) where A, B, and C are continuously differentiable functions of a single variable. Substituting u in the original PDE. and I.C. we find A, B, and C:

\[ x^2 = u(x,0) = A(0)x^2 + B(0)x + C(0) \quad \text{for all } -\infty < x < \infty , \]

\[ 0 = u_t - ku_{xx} = A(tx^2 + Bt)x + C(t) - 2kA(t) \quad \text{for } -\infty < x < \infty , 0 < t < \infty . \]

From the first equations we obtain \( A(0) = 1 \) and \( B(0) = C(0) = 0. \)

From the second equations, it follows that \( A(t) = B(t) = 0 \) and \( C(t) = 2kA(t) \)

for all \( 0 < t < \infty . \) Therefore

\[ A(t) = \text{constant} = A(0) = 1, \]

\[ B(t) = \text{constant} = B(0) = 0, \]

\[ \text{and} \quad C(t) = 2kt; \]

that is \( u(x,t) = x^2 + 2kt. \)

---

#10 (a) Solve exercise #9 using the general formula

\[ u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4kt}} e^{-\frac{(x-y)^2}{4kt}} q(y) \, dy \]

discussed in the text. This expresses \( u = u(x,t) \) as a certain integral.

Substitute \( p = \frac{x-y}{\sqrt{4kt}} \) in this integral.

(b) Since the solution is unique (see exercise #9), the resulting formula must agree with the answer to exercise #9.

Deduce the value of \( \int_{-\infty}^{\infty} p^2 e^{-p^2} \, dp. \)
Sec. 2.4, pp. 50-52.

#10 (cont.)

\[ u(x, t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/(4kt)} \, dy \quad \text{set } p = \frac{x-y}{\sqrt{4kt}}. \]

Then \( dp = \frac{dy}{\sqrt{4kt}} \).

\[
= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-p^2} \left( \frac{-4ktp + x}{\sqrt{4kt}} \right)^2 \, dp
= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-p^2} \left( 4kt^2 - 2xp\sqrt{4kt} + x^2 \right) \, dp
= \frac{4kt}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2} \, dp - \frac{2x\sqrt{4kt}}{\sqrt{\pi}} \int_{-\infty}^{\infty} p e^{-p^2} \, dp + \frac{x^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} \, dp.
\]

Therefore \( u(x, t) = x^2 + \frac{4kt}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2} \, dp \) for \( -\infty < x < \infty, \ 0 < t < \infty \).

Comparing this with the formula for \( u \) in \#9, we have \( u(x, t) = x^2 + 2kt \).

Consequently, equating the coefficients of \( x^2 \) and \( t \) we find \( 1 = 1 \) and

\[
\frac{4k}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2} \, dp = 2k. \quad \text{Thus}
\]

\[
\int_{-\infty}^{\infty} p^2 e^{-p^2} \, dp = \frac{\sqrt{\pi}}{2^2}.
\]

Note: This formula can also be deduced by integration by parts and exercise \#7:

\[
\int_{-\infty}^{\infty} p^2 e^{-p^2} \, dp = -\frac{p^2 e^{-p^2}}{2} \bigg|_{-\infty}^{\infty} + \frac{1}{2} \int_{-\infty}^{\infty} e^{-p^2} \, dp = 0 + \frac{\sqrt{\pi}}{2}.
\]

( \( u = p \), \( dv = pe^{p^2} \, dp \) )
Sec. 2.4, pp. 50-52.

14. Let \( \varphi \) be a continuous function such that \( |\varphi(x)| \leq C e^{x^2} \) for \( -\infty < x < \infty \).

Show that the formula

\[
    u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y) \, dy
\]

for solutions to \( t - ku_{xx} = 0 \) on \( -\infty < x < \infty, \, 0 < t < \infty \) satisfying \( u(x,0) = \varphi(x) \) for \( -\infty < x < \infty \), makes sense for \( 0 < t < \frac{1}{4k} \), but not necessarily for larger \( t \).

\[
    |u(x,t)| \leq \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} |\varphi(y)| \, dy \\
    \leq \frac{C}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cdot e^{y^2} \, dy \\
    = \frac{C}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{x^2 + 2xy - y^2 + 4kty^2}{4kt}} \, dy \\
    = \frac{C}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{x^2 + 2xy - (1-4kt)y^2}{4kt}} \, dy
\]

In order to guarantee convergence of the improper integral, we must have the coefficient of \( y^2 \) be negative. I.e., \( 1-4kt > 0 \) and thus \( (0 <) t < \frac{1}{4k} \). Clearly, if \( \varphi(x) = C e^{x^2} \) for \( -\infty < x < \infty \) and if \( t = \frac{1}{4k} \), the above analysis shows that the improper integral defining \( u \) diverges \((t \to +\infty)\).
Sec. 2.4, pp. 50-52

#15 Prove the uniqueness (of solutions) of the diffusion problem with Neumann boundary conditions:

\[
\begin{align*}
 u_t - ku_{xx} &= f(x, t) \quad \text{for } 0 < x < L \text{ and } 0 < t < \infty, \\
 u(x, 0) &= g(x) \quad \text{for } 0 \leq x \leq L, \\
 u_x(0, t) &= q(t) \text{ and } u_x(L, t) = r(t) \quad \text{for } 0 \leq t < \infty,
\end{align*}
\]

(*).

By the energy method.

(We use the technique of proof given on p. 43.) Let \( u = u_1(x, t) \) and \( v = u_2(x, t) \) be solutions to (*). Consider \( W(x, t) = u_1(x, t) - u_2(x, t) \); then \( W \) is a solution to

\[
\begin{align*}
 W_t - kW_{xx} &= 0 \quad \text{for } 0 < x < L \text{ and } 0 < t < \infty, \\
 W(x, 0) &= 0 \quad \text{for } 0 \leq x \leq L, \\
 W_x(0, t) &= W_x(L, t) = 0 \quad \text{for } 0 \leq t < \infty.
\end{align*}
\]

Thus

\[
0 = 0 \cdot W = (W_t - kW_{xx})W = WW_t - kW_{xx}W.
\]

But applying standard differentiation facts gives

\[
\left( \frac{1}{2} W^2 \right)_t = WW_t
\]

\[
(W_x W)_x = W_{xx} W + W_x W_x \quad \Rightarrow \quad W_{xx} W = (W_x W)_x - (W_x)^2.
\]

Substituting these expressions into (***) yields

\[
0 = \left( \frac{1}{2} W^2 \right)_t - k(W_x W)_x + k(W_x)^2.
\]
Integrating (**) in the interval 0 < x < L produces

\[ 0 = \int_0^L \left( \frac{1}{2} w^2(x,t) \right) dx - k \int_0^L \left[ w_x(x,t) w(x,t) \right] dx + k \int_0^L \left[ w_x(x,t) \right]^2 dx, \]

and then,

\[ \frac{d}{dt} \left[ \frac{1}{2} \int_0^L w^2(x,t) dx \right] = \int_0^L \left[ \frac{1}{2} w^2(x,t) \right]_t dx \quad \text{(see Theorem 1, A.3, p. 390)} \]

\[ = k \int_0^L \left[ w_x(x,t) w(x,t) \right] dx - k \int_0^L \left[ w_x(x,t) \right]^2 dx \]

\[ = k \left( w_x(L,t) w(L,t) - w_x(0,t) w(0,t) \right) - k \int_0^L \left[ w_x(x,t) \right]^2 dx \]

\[ = -k \int_0^L \left[ w_x(x,t) \right]^2 dx \]

by the Neumann boundary conditions in (†). That is, the function

(***) \quad E(t) = \frac{1}{2} \int_0^L w^2(x,t) dx

is nonincreasing since its derivative, \(-k \int_0^L \left[ w_x(x,t) \right]^2 dx\), is negative.

Therefore, if \( t > 0 \) then

\[ E(t) \leq E(0) = \frac{1}{2} \int_0^L w^2(x,0) dx = \frac{1}{2} \int_0^L 0 dx = 0. \]

But clearly \( 0 \leq E(t) \), so \( E(t) = 0 \) for all \( t > 0 \). Consequently, \( w(x,t) = 0 \) for all \( 0 \leq x \leq L \) and \( 0 \leq t < \infty \) (see (****)) and the Vanishing Theorem, A.1, p. 385), and thus \( w(x,t) = 0 \). It follows that \( u_1(x,t) - u_2(x,t) = w(x,t) = 0 \) for all \( 0 \leq x \leq L, 0 \leq t < \infty \); that is, solutions to (§) are unique.
Solve the diffusion equation with constant dissipation:

\[ u_t - ku_{xx} + bu = 0 \quad \text{for} \ -\infty < x < \infty, \ 0 < t < \infty, \]

with \( u(x,0) = \varphi(x) \) for \(-\infty < x < \infty\), where \( b > 0 \) is a constant.

Let \( u(x,t) = e^{-bt}v(x,t) \). Then \( u_t = -be^{-bt}v + e^{-bt}v_t \) and \( u_{xx} = e^{-bt}v_{xx} \), so the original PDE becomes

\[ -be^{-bt}v + e^{-bt}v_t - ke^{-bt}v_{xx} + be^{-bt}v = 0. \]

Thus \( v = v(x,t) \) solves

\[
\begin{align*}
v_t - kv_{xx} &= 0 \quad \text{for} \ -\infty < x < \infty, \ 0 < t < \infty, \\
v(x,0) &= \varphi(x) \quad \text{for} \ -\infty < x < \infty, 
\end{align*}
\]

The solution to this B.V.P. is

\[ v(x,t) = \frac{1}{\sqrt{4krt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y) \, dy. \]

Therefore

\[ u(x,t) = e^{-bt}v(x,t) = \frac{e^{-bt}}{\sqrt{4krt}} \int_{-\infty}^{\infty} e^{\frac{(x-y)^2}{4kt}} \varphi(y) \, dy. \]
Sec. 2.4, pp. 50-52.

*18. Solve the heat equation with convection: \( u_t - ku_{xx} + Vu_x = 0 \) for \(-\infty < x < \infty \) and \( 0 < t < \infty \) with \( u(x,0) = \varphi(x) \) for \(-\infty < x < \infty \), where \( V \) is a constant.

We make the following change of independent variables:

\[
\begin{cases}
\tau = t, \\
y = x - Vt.
\end{cases}
\]

If \( u = u(x,t) = u(x(y,\tau), t(y,\tau)) = V(y,\tau) \) solves the heat equation with convection, then by the chain rule and (†)

\[
V\tau = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \tau} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial \tau} = V u_x + u_t,
\]

\[
V_y = \frac{\partial u}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial y} = u_x,
\]

and \( V_{yy} = u_{xx} \).

Substituting these expressions into the heat equation with convection yields

\[
0 = u_t - ku_{xx} + Vu_x = V\tau - kV_{yy} \quad \text{for} \quad -\infty < y < \infty, \quad 0 < \tau < \infty, \]

and the original initial condition becomes

\[
(*) \quad V(y,0) = \varphi(y) \quad \text{for} \quad -\infty < y < \infty
\]

[since \( y = x \) when \( \tau = 0 \)]. The solution to (*)-(***) is

\[
V(y,\tau) = \frac{1}{\sqrt{4\pi k\tau}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4k\tau}} \varphi(z) \, dz.
\]

Using (†) we have that the solution to the heat equation with convection is

\[
u(x,t) = V(\sqrt{kt}, \frac{x}{Vt}) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-Vt-z)^2}{4kt}} \varphi(z) \, dz.
\]