Sec. 2.4, pp. 50-52.

#1. Solve the diffusion equation $u_t - ku_{xx} = 0$ for $-\infty < x < \infty$, $0 < t < \infty$, with the initial condition $u(x,0) = \varphi(x)$ for $-\infty < x < \infty$ where

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| < l, \\ 0 & \text{if } |x| > l. \end{cases}$$

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)\,dy$$

$$= \int_{-l}^{l} \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}} dy \longleftarrow \text{Let } p = \frac{x-y}{\sqrt{4kt}}. \text{ Then } dp = \frac{-dy}{\sqrt{4kt}}$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{x-l}{\sqrt{4kt}}}^{\frac{x+l}{\sqrt{4kt}}} e^{-p^2} dp$$

$$= \frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \left[\int_0^{\frac{x+l}{\sqrt{4kt}}} - \int_0^{\frac{x-l}{\sqrt{4kt}}} \right] e^{-p^2} dp$$

$$= \frac{1}{2}\left(\mathrm{Erf}\left(\frac{x+l}{\sqrt{4kt}}\right) - \mathrm{Erf}\left(\frac{x-l}{\sqrt{4kt}}\right)\right)$$

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#3 Solve $u_t - ku_{xx} = 0$ for $-\infty < x < \infty$, $0 < t < \infty$,

given that $u(x,0) = \varphi(x) = e^{3x}$ for $-\infty < x < \infty$.

By (8), p.48 the solution is

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)\,dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{x^2-2xy+y^2}{4kt}} \cdot e^{3y}\,dy$$

Add exponents. Complete square in y variable.

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{y^2-2y(x+6kt)+(x+6kt)^2-12xkt-36k^2t^2}{4kt}}\,dy$$

$$= \frac{e^{\frac{+12xkt+36k^2t^2}{4kt}}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-(x+6kt))^2}{4kt}}\,dy$$

$$\begin{cases} \text{Let } u = \dfrac{y-(x+6kt)}{\sqrt{4kt}} \\ \text{Then } du = dy/\sqrt{4kt}. \end{cases}$$

$$= \frac{e^{3x+9kt}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2}\,du$$

$$= \boxed{e^{3x+9kt}} \quad \text{(by exercise \#7, p.51)}$$

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#5. Prove properties (a) to (e) of (solutions to) the diffusion equation $u_t - ku_{xx} = 0$, $-\infty < x < \infty$, $0 < t < \infty$.

(a) The translate $v(x,t) = u(x-y,t)$ of any solution $u = u(x,t)$ is another solution, for any fixed y.

(b) Any derivative, $v(x,t) = u_x(x,t)$ or $w(x,t) = u_t(x,t)$, of a solution $u = u(x,t)$ is again a solution. (Of course, we assume that the solution u is sufficiently differentiable.)

(c) A linear combination $w(x,t) = c_1 u(x,t) + c_2 v(x,t)$ of solutions $u = u(x,t)$ and $v = v(x,t)$ is again a solution.

(d) If $S = S(x,t)$ is a solution and $g = g(y)$ is "any" function of a single variable then

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)\,dy$$

is a solution, as long as this improper integral converges "appropriately".

(e) If $u = u(x,t)$ is a solution and $a > 0$ is a constant then the dilated function $v = u(\sqrt{a}\,x, at)$ is a solution.

(a) $$v_t - kv_{xx} = u_t(x-y,t) - ku_{xx}(x-y,t) = 0 .$$

(b) $$v_t - kv_{xx} = u_{xt} - ku_{xxx} = (u_t - ku_{xx})_x = (\,0\,)_x = 0.$$

$$w_t - kw_{xx} = u_{tt} - k u_{txx} = (u_t - ku_{xx})_t = (\,0\,)_t = 0 .$$

(c) $$w_t - kw_{xx} = (c_1 u + c_2 v)_t - k(c_1 u + c_2 v)_{xx} = c_1(u_t - ku_{xx}) + c_2(v_t - kv_{xx}) = 0 .$$

(d) $$v_t - kv_{xx} = \int_{-\infty}^{\infty} S_t(x-y,t)g(y)\,dy - k\int_{-\infty}^{\infty} S_{xx}(x-y,t)g(y)\,dy$$

$$= \int_{-\infty}^{\infty} \Big[\overbrace{S_t(x-y,t) - k S_{xx}(x-y,t)}^{0}\Big] g(y)\,dy = 0 .$$

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#5 (cont.) $\quad v(x,t) = u(\overbrace{\sqrt{a}\,x}^{\xi}, \overbrace{at}^{\eta})$

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial \xi}\overset{\sqrt{a}}{\frac{\partial \xi}{\partial x}} + \frac{\partial u}{\partial \eta}\overset{0}{\frac{\partial \eta}{\partial x}} = \sqrt{a}\, u_x(\sqrt{a}\,x, at)$$

$$\frac{\partial^2 v}{\partial x^2} = \sqrt{a}\left(\frac{\partial u_x}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial x}\right) = a u_{xx}(\sqrt{a}\,x, at)$$

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial \xi}\overset{0}{\frac{\partial \xi}{\partial t}} + \frac{\partial u}{\partial \eta}\overset{a}{\frac{\partial \eta}{\partial t}} = a u_t(\sqrt{a}\,x, at)$$

Therefore
$$\begin{aligned} v_t(x,t) - k v_{xx}(x,t) &= a u_t(\sqrt{a}\,x, at) - a u_{xx}(\sqrt{a}\,x, at) \\ &= a(u_t - k u_{xx})(\sqrt{a}\,x, at) \\ &= 0. \end{aligned}$$

#9. Solve the diffusion equation $u_t - k u_{xx} = 0$ for $-\infty < x < \infty$, $0 < t < \infty$, with the initial condition $u(x,0) = x^2$ for $-\infty < x < \infty$.

Let $v(x,t) = u_{xxx}(x,t)$. Then

$$v_t - k v_{xx} = u_{xxxt} - k u_{xxxxx} = (u_t - k u_{xx})_{xxx} = (0)_{xxx} = 0$$

and
$$v(x,0) = u_{xxx}(x,0) = \frac{d^3}{dx^3}(x^2) = 0.$$

By a uniqueness theorem (for example, see F. John, Partial Differential Equations (4th ed.), pp. 216–218), it follows that the only solution of this initial value problem which obeys $v(x,t) \le M e^{ax^2}$ for all $-\infty < x < \infty$, $0 < t < \infty$, is the trivial solution $v(x,t) \equiv 0$.

Integrating the zero solution three times with respect to x (holding t fixed), we find that $u(x,t) = A(t)x^2 + B(t)x + C(t)$

#9 (cont.) where A, B, and C are continuously differentiable functions of a single variable. Substituting u in the original P.D.E. and I.C. we find A, B, and C:

$$x^2 = u(x,0) = A(0)x^2 + B(0)x + C(0) \quad \text{for all } -\infty < x < \infty,$$

$$0 = u_t - ku_{xx} = A'(t)x^2 + B'(t)x + C'(t) - 2kA(t) \quad \text{for } -\infty < x < \infty,\ 0 < t < \infty.$$

From the first equations we obtain $A(0) = 1$ and $B(0) = C(0) = 0$.
From the second equations, it follows that $A'(t) = B'(t) = 0$ and $C'(t) = 2kA(t)$ for all $0 < t < \infty$. Therefore

$$A(t) = \text{constant} = A(0) = 1,$$
$$B(t) = \text{constant} = B(0) = 0,$$

and $C(t) = 2kt$;

that is $\boxed{u(x,t) = x^2 + 2kt}$.

#10 (a) Solve exercise #9 using the general formula

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi t}}\, e^{-\frac{(x-y)^2}{4kt}}\, \varphi(y)\,dy$$

discussed in the text. This expresses $u = u(x,t)$ as a certain integral. Substitute $p = \frac{x-y}{\sqrt{4kt}}$ in this integral.

(b) Since the solution is unique (see exercise #9), the resulting formula must agree with the answer to exercise #9. Deduce the value of $\int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp$.

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#10 (cont.)

(a) $$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}} y^2 \, dy \leftarrow \text{Let } p = \frac{x-y}{\sqrt{4kt}}.$$

Then $dp = \frac{-dy}{\sqrt{4kt}}$.

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-p^2} \left(-\sqrt{4kt}\,p + x\right)^2 dp$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-p^2} \left(4ktp^2 - 2xp\sqrt{4kt} + x^2\right) dp$$

$$= \frac{4kt}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2} dp - \frac{2x\sqrt{4kt}}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} p e^{-p^2} dp}_{0 \text{ (oddness!)}} + \frac{x^2}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-p^2} dp}_{\sqrt{\pi}}.$$

Therefore $u(x,t) = x^2 + \frac{4kt}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2} dp$ for $-\infty < x < \infty$, $0 < t < \infty$.

Comparing this with the formula for u in #9, we have $u(x,t) = x^2 + 2kt$.

Consequently, equating the coefficients of x^2 and t we find $1 = 1$ and $\frac{4k}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2} dp = 2k$. Thus

$$\boxed{\int_{-\infty}^{\infty} p^2 e^{-p^2} dp = \frac{\sqrt{\pi}}{2}}.$$

Note: This formula can also be deduced by integration by parts and exercise #7:

$$\int_{-\infty}^{\infty} p^2 e^{-p^2} dp = -\frac{p}{2} e^{-p^2} \Big|_{-\infty}^{\infty} + \frac{1}{2} \int_{-\infty}^{\infty} e^{-p^2} dp = 0 + \frac{\sqrt{\pi}}{2}.$$

$(u = p,\ dv = pe^{-p^2} dp)$

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#14. Let φ be a continuous function such that $|\varphi(x)| \leq Ce^{x^2}$ for $-\infty < x < \infty$. Show that the formula

$$(8) \qquad u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)\,dy$$

for solutions to $u_t - ku_{xx} = 0$ on $-\infty < x < \infty$, $0 < t < \infty$, satisfying $u(x,0) = \varphi(x)$ for $-\infty < x < \infty$, makes sense for $0 < t < \frac{1}{4k}$, but not necessarily for larger t.

$$|u(x,t)| \leq \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} |\varphi(y)|\,dy$$

$$\leq \frac{C}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cdot e^{y^2}\,dy$$

$$= \frac{C}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{\frac{-x^2+2xy-y^2+4kty^2}{4kt}}\,dy$$

$$= \frac{C}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{\frac{-x^2+2xy-(1-4kt)y^2}{4kt}}\,dy$$

In order to guarantee convergence of the improper integral, we must have the coefficient of y^2 be negative. I.e. $1-4kt > 0$ and thus $(0<)\,t < \frac{1}{4k}$. Clearly, if $\varphi(x) = Ce^{x^2}$ for $-\infty < x < \infty$ and if $t \geq \frac{1}{4k}$, the above analysis shows that the improper integral defining u diverges (to $+\infty$).

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#15 Prove the uniqueness (of solutions) of the diffusion problem with Neumann boundary conditions:

$$(*)\quad \begin{cases} u_t - ku_{xx} = f(x,t) & \text{for } 0<x<L \text{ and } 0<t<\infty, \\ u(x,0) = \phi(x) & \text{for } 0\le x\le L, \\ u_x(0,t) = g(t) \text{ and } u_x(L,t) = h(t) & \text{for } 0\le t<\infty, \end{cases}$$

by the energy method.

(We use the technique of proof given on p.43.) Let $u = u_1(x,t)$ and $u = u_2(x,t)$ be solutions to $(*)$. Consider $w(x,t) = u_1(x,t) - u_2(x,t)$; then w is a solution to

$$(\dagger)\quad \begin{cases} w_t - kw_{xx} = 0 & \text{for } 0<x<L \text{ and } 0<t<\infty, \\ w(x,0) = 0 & \text{for } 0\le x\le L, \\ w_x(0,t) = w_x(L,t) = 0 & \text{for } 0\le t<\infty. \end{cases}$$

Thus

$$(**)\quad 0 = 0\cdot w = (w_t - kw_{xx})w = ww_t - kw_{xx}w.$$

But applying standard differentiation facts gives

$$\left(\tfrac{1}{2}w^2\right)_t = ww_t$$

$$(w_x w)_x = w_{xx}w + w_x w_x \quad\Rightarrow\quad w_{xx}w = (w_x w)_x - (w_x)^2.$$

Substituting these expressions into $(**)$ yields

$$(***)\quad 0 = \left(\tfrac{1}{2}w^2\right)_t - k(w_x w)_x + k(w_x)^2.$$

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#15 (cont.) Integrating (***) in the interval $0<x<L$ produces

$$0 = \int_0^L \left(\tfrac{1}{2}w^2(x,t)\right)_t dx - k\int_0^L \left[w_x(x,t)w(x,t)\right]_x dx + k\int_0^L \left[w_x(x,t)\right]^2 dx,$$

and thus,

$$\frac{d}{dt}\left[\frac{1}{2}\int_0^L w^2(x,t)\,dx\right] = \int_0^L \left[\tfrac{1}{2}w^2(x,t)\right]_t dx \qquad \text{(see Theorem 1, A.3, p. 390)}$$

$$= k\int_0^L \left[w_x(x,t)w(x,t)\right]_x dx - k\int_0^L \left[w_x(x,t)\right]^2 dx$$

$$= k\left(w_x(L,t)w(L,t) - w_x(0,t)w(0,t)\right) - k\int_0^L \left[w_x(x,t)\right]^2 dx$$

$$= -k\int_0^L \left[w_x(x,t)\right]^2 dx$$

by the Neumann boundary conditions in (†). That is, the function

$$(\ast\ast\ast\ast) \qquad E(t) = \frac{1}{2}\int_0^L w^2(x,t)\,dx$$

is nonincreasing since its derivative, $-k\int_0^L [w_x(x,t)]^2 dx$, is nonpositive. Therefore, if $t>0$ then

$$E(t) \le E(0) = \frac{1}{2}\int_0^L w^2(x,0)\,dx \overset{\text{by } (\dagger)}{=} \frac{1}{2}\int_0^L 0\,dx = 0.$$

But clearly $0 \le E(t)$, so $E(t) = 0$ for all $t \ge 0$. Consequently $w^2(x,t) = 0$ for all $0 \le x \le L$ and $0 \le t < \infty$ (see (****) and the Vanishing Theorem, A.1, p. 385), and thus $w(x,t) = 0$. It follows that $u_1(x,t) - u_2(x,t) = w(x,t) = 0$ for all $0 \le x \le L$, $0 \le t < \infty$; that is, solutions to (*) are unique.

Sec. 2.4, pp. 50–52.

#16. Solve the diffusion equation with constant dissipation:

$$u_t - ku_{xx} + bu = 0 \quad \text{for } -\infty < x < \infty,\ 0 < t < \infty,$$

with $u(x,0) = \varphi(x)$ for $-\infty < x < \infty$, where $b > 0$ is a constant.

Let $u(x,t) = e^{-bt} v(x,t)$. Then $u_t = -be^{-bt} v + e^{-bt} v_t$ and $u_{xx} = e^{-bt} v_{xx}$, so the original PDE becomes

$$\cancel{-be^{-bt} v} + e^{-bt} v_t - ke^{-bt} v_{xx} + \cancel{be^{-bt} v} = 0.$$

Thus $v = v(x,t)$ solves

$$\begin{cases} v_t - kv_{xx} = 0 & \text{for } -\infty < x < \infty,\ 0 < t < \infty, \\ v(x,0) = \varphi(x) & \text{for } -\infty < x < \infty. \end{cases}$$

The solution to this B.V.P. is

$$v(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)\,dy.$$

Therefore

$$u(x,t) = e^{-bt} v(x,t) = \frac{e^{-bt}}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)\,dy.$$

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#18. Solve the heat equation with convection: $u_t - ku_{xx} + Vu_x = 0$ for $-\infty < x < \infty$ and $0 < t < \infty$ with $u(x,0) = \varphi(x)$ for $-\infty < x < \infty$, where V is a constant.

We make the following change of independent variables:

$$(\dagger) \quad \begin{cases} \tau = t, \\ y = x - Vt. \end{cases}$$

If $u = u(x,t) = u(x(y,\tau), t(y,\tau)) = v(y,\tau)$ solves the heat equation with convection, then by the chain rule and $(\dagger)$

$$v_\tau = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \tau} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial \tau} = Vu_x + u_t,$$

$$v_y = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y} = u_x,$$

and $v_{yy} = u_{xx}$.

Substituting these expressions into the heat equation with convection yields

$$(*) \qquad 0 = u_t - ku_{xx} + Vu_x = v_\tau - kv_{yy} \quad \text{for } -\infty < y < \infty,\ 0 < \tau < \infty,$$

and the original initial condition becomes

$$(**) \qquad v(y,0) = \varphi(y) \quad \text{for } -\infty < y < \infty$$

[since $y = x$ when $t = 0$]. The solution to $(*)$–$(**)$ is

$$v(y,\tau) = \frac{1}{\sqrt{4\pi k\tau}} \int_{-\infty}^{\infty} e^{-\frac{(y-z)^2}{4k\tau}} \varphi(z)\,dz.$$

Using $(\dagger)$ we have that the solution to the heat equation with convection is

$$\boxed{u(x,t) = v(\overset{y}{x - Vt}, \overset{\tau}{t}) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-Vt-z)^2}{4kt}} \varphi(z)\,dz.}$$