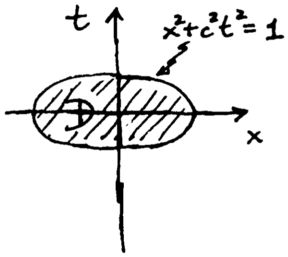
Sec. 2.5, p.54.

\# 1. Show that there is no maximum principle for the wave equation.

Let $u(x,t) = -(x^2 + c^2t^2)$ on the region $\bar{D}$: $x^2 + c^2t^2 \le 1$.



Then $u_{tt} - c^2 u_{xx} = -2c^2 - c^2(-2) = 0$ at all points of D, and at each point (x_0, t_0) on the boundary of D we have $x_0^2 + c^2 t_0^2 = 1$ so $u(x_0, t_0) = -1$.

However $u(0,0) = 0$ so the maximum of $u = u(x,t)$ on $\bar{D}$ does not occur at any boundary point of $\bar{D}$.

#2. Here is a direct relationship between the wave and diffusion equations. Let $u = u(x,t)$ solve the wave equation on the whole line with bounded second derivatives. Let

$$(*) \qquad v(x,t) = \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{s^2c^2}{4kt}} u(x,s)\,ds.$$

(a) Show that $v = v(x,t)$ solves the diffusion equation on $-\infty<x<\infty$, $t>0$.

(b) Show that $\lim_{t\to 0^+} v(x,t) = u(x,0)$ for $-\infty<x<\infty$.

Write $H(s,t) = \dfrac{c e^{-\frac{s^2c^2}{4kt}}}{\sqrt{4k\pi t}}$, and note that $(*)$ can be written as

$$(*') \qquad v(x,t) = \int_{-\infty}^{\infty} u(x,s) H(s,t)\,ds.$$

For solving (a), we use the following results which are proved in an appendix at the end of this problem.

<u>FACT 1</u>: $H = H(s,t)$ solves a diffusion equation

$$H_t - \frac{k}{c^2} H_{ss} = 0$$

on $-\infty < s < \infty$, $t>0$.

<u>FACT 2</u>: $\lim_{s\to\pm\infty} u(x,s) H_s(s,t) = \lim_{s\to\pm\infty} u_s(x,s) H(s,t) = 0$ for all $-\infty<x<\infty$, $t>0$.

(a) Using Theorem 2, A.3, p.390, we have upon differentiating $(*')$:

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,s)H(s,t)\,ds = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left[u(x,s)H(s,t)\right]ds =$$

#2(a) (cont.)

$$= \int_{-\infty}^{\infty} u(x,s)\, H_t(s,t)\, ds \overset{\text{FACT 1}}{=} \int_{-\infty}^{\infty} \frac{k}{c^2}\, u(x,s)\, H_{ss}(s,t)\, ds .$$

Similarly, $\dfrac{\partial v}{\partial x} = \displaystyle\int_{-\infty}^{\infty} u_x(x,s)\, H(s,t)\, ds$ and $\dfrac{\partial^2 v}{\partial x^2} = \displaystyle\int_{-\infty}^{\infty} u_{xx}(x,s)\, H(s,t)\, ds$.

Integrating by parts twice the integral expression for $\frac{\partial v}{\partial t}$ and applying FACT 2 to the boundary terms, gives

$$(**) \qquad \frac{\partial v}{\partial t} = \int_{-\infty}^{\infty} \frac{k}{c^2}\, \overbrace{H_{ss}(s,t)}^{dv}\, \overbrace{u(x,s)}^{v}\, \overbrace{ds}$$

$$= \lim_{\substack{s_1 \to \infty \\ s_2 \to -\infty}} \frac{k}{c^2}\, u(x,s)\, H_s(s,t) \Big|_{s=s_2}^{s=s_1} - \int_{-\infty}^{\infty} \frac{k}{c^2}\, u_s(x,s)\, H_s(s,t)\, ds$$

$$= -\frac{k}{c^2} \int_{-\infty}^{\infty} \overbrace{u_s(x,s)}^{v}\, \overbrace{H_s(s,t)\, ds}^{dv}$$

$$= -\frac{k}{c^2} \lim_{\substack{s_1 \to \infty \\ s_2 \to -\infty}} u_s(x,s)\, H(s,t) \Big|_{s=s_2}^{s=s_1} + \frac{k}{c^2} \int_{-\infty}^{\infty} H(s,t)\, u_{ss}(x,s)\, ds$$

$$= \frac{k}{c^2} \int_{-\infty}^{\infty} H(s,t)\, u_{ss}(x,s)\, ds .$$

Because u satisfies the wave equation

$$u_{ss}(x,s) - c^2 u_{xx}(x,s) = 0$$

for $-\infty < x < \infty$ and $-\infty < s < \infty$, we have

$$(***) \qquad \frac{\partial^2 v}{\partial x^2} = \int_{-\infty}^{\infty} u_{xx}(x,s)\, H(s,t)\, ds = \frac{1}{c^2} \int_{-\infty}^{\infty} H(s,t)\, u_{ss}(x,s)\, ds .$$

#2(a) (cont.) From (**) and (***), it follows that

$$\frac{\partial v}{\partial t} - k\frac{\partial^2 v}{\partial x^2} = 0 \qquad \text{for } -\infty < x < \infty \ , \quad t > 0 .$$

(b) Fix $x_0 \in (-\infty, \infty)$. In order to show that

$$\lim_{t \to 0^+} v(x_0, t) = u(x_0, 0),$$

we will make use of the fact that if φ is a continuous function on $(-\infty, \infty)$ with bounded second derivative then

$$(****) \qquad w(\xi, t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(\xi - z)^2}{4kt}}}{\sqrt{4\pi k t}} \varphi(z)\, dz$$

is the solution to the diffusion equation

$$w_t - k w_{\xi\xi} = 0 \qquad \text{for } -\infty < \xi < \infty \ , \quad t > 0,$$

satisfying $\displaystyle\lim_{t \to 0^+} w(\xi, t) \overset{(*****)}{=} \varphi(\xi)$ for $-\infty < \xi < \infty$. (See theorem 1, Sec. 3.5, p. 79; the hypothesis that φ is bounded on $(-\infty,\infty)$ can be replaced by φ'' is bounded on $(-\infty,\infty)$. The proof requires more careful integral estimates than pp. 79-80.)

Let $S(x,t) = \dfrac{e^{-\frac{x^2}{4kt}}}{\sqrt{4\pi k t}}$; note that (****) is equivalent to

$$(****') \qquad w(\xi, t) = \int_{-\infty}^{\infty} S(\xi - z, t)\varphi(z)\, dz .$$

Also observe that $H(s,t) = c\, S(sc, t)$ for $-\infty < s < \infty$, $t > 0$.

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#2 (b) (cont.) Therefore

$$v(x_0,t) = \int_{-\infty}^{\infty} u(x_0,s)\,H(s,t)\,ds$$

$$= \int_{-\infty}^{\infty} c\,S(sc,t)\,u(x_0,s)\,ds$$

Let $z = sc$

then $dz = c\,ds$

$$= \int_{-\infty}^{\infty} S(z,t)\,\overbrace{u(x_0,\tfrac{z}{c})}^{\varphi(z)}\,dz.$$

Thus

$$\lim_{t\to 0^+} v(x_0,t) = \lim_{t\to 0^+} \int_{-\infty}^{\infty} S(z-\underset{\zeta=0}{0},t)\,\overbrace{u(x_0,\tfrac{z}{c})}^{\varphi(z)}\,dz$$

$$= \varphi(\zeta)\Big|_{\zeta=0} \qquad \text{(by (****$'$) and (*****))}$$

$$= \varphi(0)$$

$$= u(x_0,\tfrac{0}{c})$$

$$= u(x_0,0).$$

Appendix.

Proof of FACT 1: It is easy to check that $S(x,t) = e^{-\frac{x^2}{4kt}}/\sqrt{4\pi kt}$ solves $S_t - k\,S_{xx} = 0$ on $-\infty < x < \infty$, $t > 0$. Since $H(s,t) = c\,S(\overset{x}{sc},t)$,

$\frac{\partial H}{\partial s} = c\,\frac{\partial S}{\partial x}\cdot\frac{\partial x}{\partial s} = c^2\frac{\partial S}{\partial x}$. Likewise $\frac{\partial^2 H}{\partial s^2} = c^2\frac{\partial}{\partial s}\left(\frac{\partial S}{\partial x}\right) = c^2\frac{\partial^2 S}{\partial x^2}\cdot\frac{\partial x}{\partial s} = c^3\frac{\partial^2 S}{\partial x^2}$. Clearly $\frac{\partial H}{\partial t} = c\,\frac{\partial S}{\partial t}$. Substituting in $S_t - k\,S_{xx} = 0$ gives

$\frac{1}{c}\frac{\partial H}{\partial t} - \frac{k}{c^3}\frac{\partial^2 H}{\partial s^2} = 0$, i.e. $H_t - \frac{k}{c^2}H_{ss} = 0$.

#2 Appendix (cont.)

Proof of FACT 2: We are given that u has bounded second derivatives; suppose M is a real number such that

$$(*) \quad |u_{tt}(x,t)| \le M, \quad |u_{xt}(x,t)| \le M, \quad \text{and} \quad |u_{xx}(x,t)| \le M$$

for all $-\infty < x < \infty$, $-\infty < t < \infty$. Let $-\infty < x_0 < \infty$ and $-\infty < t_0 < \infty$, and define

$$f(\lambda) = u(\overbrace{\lambda x_0}^{x}, \overbrace{\lambda t_0}^{t}) \qquad \text{for all } 0 \le \lambda \le 1.$$

By Taylor's theorem (see Rudin's, Principles of Mathematical Analysis (3rd ed.), Theorem 5.15, pp. 110–111) there is a number $\xi \in (0,1)$ such that

$$(\dagger) \quad f(\lambda) = f(0) + f'(0)\lambda + \frac{f''(\xi)}{2}\lambda^2.$$

But $f'(\lambda) = \dfrac{\partial u}{\partial x}\overbrace{\dfrac{dx}{d\lambda}}^{x_0} + \dfrac{\partial u}{\partial t}\overbrace{\dfrac{dt}{d\lambda}}^{t_0} = x_0 u_x + t_0 u_t$

and $f''(\lambda) = x_0\left(\dfrac{\partial(u_x)}{\partial x}\overbrace{\dfrac{dx}{d\lambda}}^{x_0} + \dfrac{\partial(u_x)}{\partial t}\overbrace{\dfrac{dt}{d\lambda}}^{t_0}\right) + t_0\left(\dfrac{\partial(u_t)}{\partial x}\overbrace{\dfrac{dx}{d\lambda}}^{x_0} + \dfrac{\partial(u_t)}{\partial t}\overbrace{\dfrac{dt}{d\lambda}}^{t_0}\right)$

$$= x_0^2 u_{xx} + 2x_0 t_0 u_{xt} + t_0^2 u_{tt}.$$

Therefore $u(\lambda x_0, \lambda t_0) = f(\lambda) = f(0) + f'(0)\lambda + \dfrac{f''(\xi)}{2}\lambda^2$

$$= u(0,0) + [x_0 u_x(0,0) + t_0 u_t(0,0)]\lambda$$

$$+ \left[x_0^2 u_{xx}(\xi x_0, \xi t_0) + 2x_0 t_0 u_{xt}(\xi x_0, \xi t_0) + t_0^2 u_{tt}(\xi x_0, \xi t_0)\right]\frac{\lambda^2}{2}.$$

In particular, when $\lambda = 1$, this becomes

$$(**) \quad u(x_0,t_0) = u(0,0) + [x_0 u_x(0,0) + t_0 u_t(0,0)] + \tfrac{1}{2}\left[x_0^2 u_{xx}(\xi x_0, \xi t_0) + 2x_0 t_0 u_{xt}(\xi x_0, \xi t_0) + \overbrace{\quad}^{t_0^2 u_{tt}(\xi x_0, \xi t_0)}\right].$$

Using $(*)$ and $(**)$, we see that for all $-\infty < x_0 < \infty$ and $-\infty < t_0 < \infty$,

#2 Appendix (cont.)

$$(***) \quad |u(x_0,t_0)| \le A + B|x_0| + C|t_0| + \frac{M}{2}\left(x_0^2 + 2|x_0||t_0| + |t_0|^2\right)$$

where $A = |u(0,0)|$, $B = |u_x(0,0)|$, $C = |u_t(0,0)|$. Note that (***) says that u does not exceed a quadratic polynomial in the variables x and t in the x-t plane.

By a similar argument, there exists a positive constant

$$(****) \quad |u_t(x_0,t_0)| \le D + M(|x_0| + |t_0|)$$

for $-\infty < x_0 < \infty$ and $-\infty < t_0 < \infty$.

We now use the estimates (***) and (****) to prove FACT 2. Fix $-\infty < x < \infty$ and $<t<\infty$. By (****) there is a constant M' such that $|u(x,s)| \le M's^2$ for all sufficiently large $|s|$. Therefore

$$|u(x,s)H_s(s,t)| = \left|u(x,s)\left(\frac{-2sc^3 e^{\frac{-s^2c^2}{4kt}}}{4kt\sqrt{4\pi kt}}\right)\right|$$

$$\le \frac{M's^2 \cdot 2|s|c^3 e^{\frac{-s^2c^2}{4kt}}}{4kt\sqrt{4\pi kt}}$$

$$= \frac{2}{\sqrt{\pi}} M' \left(\frac{|s|c}{\sqrt{4kt}}\right)^3 e^{-\left(\frac{|s|c}{\sqrt{4kt}}\right)^2}$$

$$= \frac{2}{\sqrt{\pi}} M' \lambda^3 e^{-\lambda^2} \quad \text{where } \lambda = \frac{|s|c}{\sqrt{4kt}}.$$

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#2 Appendix (cont.) Therefore

$$0 \le \lim_{|s| \to \infty} \left| u(x,s) H_s(s,t) \right| \le \lim_{\lambda \to \infty} \left(\frac{2M'}{\sqrt{\pi}} \lambda^3 e^{-\lambda^2} \right)$$

$$= \frac{2M'}{\sqrt{\pi}} \lim_{\lambda \to \infty} \left(\frac{\lambda^3}{e^{\lambda^2}} \right)$$

$$= \frac{2M'}{\sqrt{\pi}} \lim_{\lambda \to \infty} \left(\frac{3\lambda^2}{2\lambda e^{\lambda^2}} \right) \qquad \text{(l'Hospital's rule)}$$

$$= \frac{3M'}{\sqrt{\pi}} \lim_{\lambda \to \infty} \left(\frac{\lambda}{e^{\lambda^2}} \right) \qquad \text{(algebra)}$$

$$= \frac{3M'}{\sqrt{\pi}} \lim_{\lambda \to \infty} \left(\frac{1}{2\lambda e^{\lambda^2}} \right) \qquad \text{(l' Hospital's rule)}$$

$$= 0 .$$

This demonstrates the first part of FACT 2:

$$\lim_{s \to \pm\infty} u(x,s) H_s(s,t) = 0 \qquad \text{for } -\infty < x < \infty \text{ and } t > 0.$$

A similar argument using estimate (***) proves the second part:

$$\lim_{s \to \pm\infty} u_s(x,s) H(s,t) = 0 \qquad \text{for } -\infty < x < \infty \text{ and } t > 0.$$