Sec. 3.5, p. 81

#1. Prove that if φ is any bounded piecewise continuous function, then

$$\frac{1}{\sqrt{4\pi}} \int_0^{\pm\infty} e^{-p^2/4} \varphi(x + p\sqrt{kt})\,dp \to \pm \tfrac{1}{2}\varphi(x^{\pm}) \quad \text{as} \quad t \searrow 0.$$

Let $\varepsilon > 0$ and fix $x \in (-\infty, \infty)$. We must show that there exists $t_0 > 0$ such that

$$(*) \qquad \left| \frac{1}{\sqrt{4\pi}} \int_0^{\infty} e^{-p^2/4} \varphi(x + p\sqrt{kt})\,dp - \tfrac{1}{2}\varphi(x^+) \right| < \varepsilon$$

for all $t \in (0, t_0)$. Since φ is piecewise continuous at x, there exists $\delta > 0$ such that $|\varphi(y) - \varphi(x^+)| < \varepsilon$ for all $y \in (x, x+\delta)$. Let $M = \max\limits_{-\infty < y < \infty} |\varphi(y)| < \infty$. Since $\int_{-\infty}^{\infty} e^{-p^2/4}\,dp < \infty$, there exists $N > 0$ such that the "tail" of this convergent improper integral is "small", say

$$\int_N^{\infty} e^{-p^2/4}\,dp < \frac{\varepsilon\sqrt{\pi}}{2M}.$$

Set $t_0 = \dfrac{\delta^2}{kN^2}$ and suppose $t \in (0, t_0)$.

Observe first that

$$\frac{1}{\sqrt{4\pi}} \int_0^{\infty} e^{-p^2/4} \varphi(x + p\sqrt{kt})\,dp - \tfrac{1}{2}\varphi(x^+) \cdot 1$$

$$= \frac{1}{\sqrt{4\pi}} \int_0^{\infty} e^{-p^2/4} \varphi(x + p\sqrt{kt})\,dp - \tfrac{1}{2}\varphi(x^+) \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-p^2/4}\,dp$$

(Change variables in exercise #7 p. 51 to see this.)

$$= \frac{1}{\sqrt{4\pi}} \int_0^{\infty} e^{-p^2/4} \left[\varphi(x + p\sqrt{kt}) - \varphi(x^+)\right] dp.$$

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#1 (cont.) Therefore

$$\frac{1}{\sqrt{4\pi}}\int_0^\infty e^{-p^2/4}\varphi(x+p\sqrt{kt})\,dp - \frac{1}{2}\varphi(x^+) = \frac{1}{\sqrt{4\pi}}\left(\int_0^{\frac{\delta}{\sqrt{kt}}} + \int_{\frac{\delta}{\sqrt{kt}}}^{\infty}\right)e^{-p^2/4}\left[\varphi(x+p\sqrt{kt})-\varphi(x^+)\right]dp$$

$$\equiv I_1 + I_2 .$$

We estimate I_1 and I_2 as follows:

$$|I_1| \le \frac{1}{\sqrt{4\pi}}\int_0^{\frac{\delta}{\sqrt{kt}}} e^{-p^2/4}\left|\varphi(x+p\sqrt{kt})-\varphi(x^+)\right|dp$$

$$\le \left(\frac{1}{\sqrt{4\pi}}\int_0^{\frac{\delta}{\sqrt{kt}}} e^{-p^2/4}\,dp\right)\max_{y\in(x,x+\delta)}\left|\varphi(y)-\varphi(x^+)\right|$$

$$< \left(\frac{1}{\sqrt{4\pi}}\int_0^{\infty} e^{-p^2/4}\,dp\right)\cdot\varepsilon$$

$$= \frac{\varepsilon}{2}.$$

$$\left\{\begin{array}{c} 0 < p < \frac{\delta}{\sqrt{kt}} \\ \Updownarrow \\ 0 < p\sqrt{kt} < \delta \\ \Updownarrow \\ x < \underbrace{x+p\sqrt{kt}}_{y} < x+\delta \end{array}\right.$$

$$|I_2| \le \frac{1}{\sqrt{4\pi}}\int_{\frac{\delta}{\sqrt{kt}}}^{\infty} e^{-p^2/4}\left|\varphi(x+p\sqrt{kt})-\varphi(x^+)\right|dp$$

$$\le \left(\frac{1}{\sqrt{4\pi}}\int_{\frac{\delta}{\sqrt{kt}}}^{\infty} e^{-p^2/4}\,dp\right)2\max_{-\infty<y<\infty}\left|\varphi(y)\right|$$

$$\le \left(\frac{1}{\sqrt{4\pi}}\int_N^{\infty} e^{-p^2/4}\,dp\right)\cdot 2M$$

$$< \frac{1}{\sqrt{4\pi}}\cdot\frac{\varepsilon\sqrt{\pi}}{2M}\cdot 2M$$

$$= \frac{\varepsilon}{2}.$$

$$\left\{\begin{array}{c} 0 < t < t_0 = \frac{\delta^2}{kN^2} \\ \Updownarrow \\ 0 < \sqrt{kt} < \frac{\delta}{N} \\ \Downarrow \\ N < \frac{\delta}{\sqrt{kt}} \end{array}\right.$$

Therefore $(*)$ is established; i.e. $\frac{1}{\sqrt{4\pi}}\int_0^\infty e^{-p^2/4}\varphi(x+p\sqrt{kt})\,dp \to \frac{1}{2}\varphi(x^+)$ as $t \downarrow 0$.

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#1. (cont.) The proof that $\frac{1}{\sqrt{4\pi}}\int_0^{-\infty} e^{-p^2/4}\varphi(x+p\sqrt{kt})\,dp \to -\frac{1}{2}\varphi(x^-)$ as $t \searrow 0$ is completely analogous and is left to the reader.

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#2. Use exercise #1 to prove Theorem 2:

Let φ be a bounded piecewise continuous function on $(-\infty,\infty)$. Then

$$(2) \qquad u(x,t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4} \varphi(x - p\sqrt{kt})\,dp$$

is an infinitely differentiable solution of $u_t - ku_{xx} = 0$ on $-\infty < x < \infty$, $0 < t < \infty$, and

$$\lim_{t \to 0^+} u(x,t) = \frac{\varphi(x^+) + \varphi(x^-)}{2}$$

for every real x. At every point x where φ is continuous, this limit equals $\varphi(x)$.

$$\lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+} \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4} \varphi(x - p\sqrt{kt})\,dp$$

(Make the change-of-variables $p \mapsto -p$.)

$$= \lim_{t\to 0^+} \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4} \varphi(x + p\sqrt{kt})\,dp$$

$$= \lim_{t\to 0^+} \frac{1}{\sqrt{4\pi}} \int_{0}^{\infty} e^{-p^2/4} \varphi(x + p\sqrt{kt})\,dp + \lim_{t\to 0^+} \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{0} e^{-p^2/4} \varphi(x + p\sqrt{kt})\,dp$$

$$= \lim_{t\to 0^+} \frac{1}{\sqrt{4\pi}} \int_{0}^{\infty} e^{-p^2/4} \varphi(x + p\sqrt{kt})\,dp - \lim_{t\to 0^+} \frac{1}{\sqrt{4\pi}} \int_{0}^{-\infty} e^{-p^2/4} \varphi(x + p\sqrt{kt})\,dp$$

$$= \tfrac{1}{2}\varphi(x^+) - \left[-\tfrac{1}{2}\varphi(x^-)\right] \qquad \text{(by exercise \#1)}$$

$$= \frac{\varphi(x^+) + \varphi(x^-)}{2}.$$