Sec. 4.1, p. 87

#1. (a) Use the Fourier expansion to explain why the note produced by a violin string rises sharply by one octave when the string is clamped exactly at its midpoint.

(b) Explain why the note rises when the string is tightened.

(a) The violin string originally occupying $0 \le x \le \ell$ has vertical deflection u at position x and time $t \ge 0$ given by (9) on page 83:

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{\ell}\right) + B_n \sin\left(\frac{n\pi ct}{\ell}\right)\right] \sin\left(\frac{n\pi x}{\ell}\right).$$

The frequencies of the wave are the coefficients of time t:

$$\frac{n\pi c}{\ell} = \frac{n\pi}{\ell}\sqrt{\frac{T}{\rho}} \quad \text{for} \quad n = 1, 2, 3, \ldots$$

The "fundamental" note of the string occurs when $n=1$ and has frequency

$$(*) \qquad \frac{\pi}{\ell}\sqrt{\frac{T}{\rho}}.$$

If the violin string is clamped at its midpoint then the portion occupying $0 \le x \le \ell/2$ vibrates. The deflection from vertical v is given by

$$v(x,t) = \sum_{n=1}^{\infty} \left[A_n' \cos\left(\frac{n\pi ct}{\ell/2}\right) + B_n' \sin\left(\frac{n\pi ct}{\ell/2}\right)\right] \sin\left(\frac{n\pi x}{\ell/2}\right)$$

$$= \sum_{n=1}^{\infty} \left[A_n' \cos\left(\frac{2n\pi ct}{\ell}\right) + B_n' \sin\left(\frac{2n\pi ct}{\ell}\right)\right] \sin\left(\frac{2n\pi x}{\ell}\right).$$

The "fundamental" note of the shortened string has frequency

$$(**) \qquad \frac{\pi}{\ell/2}\sqrt{\frac{T}{\rho}} = \frac{2\pi}{\ell}\sqrt{\frac{T}{\rho}}$$

(cont.)

which is twice that of (*) for the original string; that is, by definition the shortened strings' fundamental note is one octave above that of the original string.

(b) From part (a), the frequencies of the vibrating violin string are

$$\frac{n\pi}{l}\sqrt{\frac{T}{\rho}} \qquad \text{where } n = 1, 2, 3, \ldots$$

If the string is tightened then the tension T increases, while ρ and l remain (approximately) constant! This causes an increase in the frequencies of the vibration; that is, the note rises in pitch.

#4. Consider waves in a resistant medium that satisfy the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} + r u_t = 0 & \text{for } 0 < x < l \text{ and } t > 0 \\ u(0,t) = u(l,t) = 0 & \text{for } t \geq 0 \\ u(x,0) = \varphi(x) \text{ and } u_t(x,0) = \psi(x) & \text{for } 0 \leq x \leq l. \end{cases}$$

Here r is a constant satisfying $0 < r < \frac{2\pi c}{l}$. Write down the series expansion of the solution.

Let $u(x,t) = X(x)T(t)$. Substituting in the PDE gives

$$T''X - c^2 X'' T + r X T' = 0.$$

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Dividing by $c^2 \mathbb{X} T$ gives

$$\frac{T''}{c^2 T} + \frac{r T'}{c^2 T} = \frac{\mathbb{X}''}{\mathbb{X}} = \text{constant} = -\lambda .$$

Thus $\begin{cases} T'' + rT' + \lambda c^2 T = 0 \\ \boxed{\mathbb{X}'' + \lambda \mathbb{X} = 0} \end{cases}$

The B.C.'s become $0 = u(0,t) = \mathbb{X}(0)T(t)$ for $t \geq 0 \Rightarrow \boxed{\mathbb{X}(0) = 0}$.
$0 = u(\ell,t) = \mathbb{X}(\ell)T(t)$ " " " $\boxed{\mathbb{X}(\ell) = 0}$.

The (only) eigenvalues for the boxed BVP are $\lambda_n = \beta_n^2 = \frac{n^2\pi^2}{\ell^2}$ $(n = 1,2,3,\dots)$.
The corresponding eigenfunctions are $\mathbb{X}_n(x) = \sin(\beta_n x) = \sin\left(\frac{n\pi x}{\ell}\right)$ $(n=1,2,3,\dots)$.
The D.E. for T becomes

$$(*) \quad T_n'' + r T_n' + \left(\frac{n\pi c}{\ell}\right)^2 T_n = 0 .$$

Substituting the solution form $T_n(t) = e^{\mu_n t}$ in this D.E. gives

$$\left[\mu_n^2 + r\mu_n + \left(\frac{n\pi c}{\ell}\right)^2\right] e^{\mu_n t} = 0 \qquad \text{for } t > 0 .$$

Hence $\mu_n^2 + r\mu_n + \left(\frac{n\pi c}{\ell}\right)^2 = 0$; the solutions are distinct and complex for any $n \geq 1$:

$$\mu_n = \frac{-r \pm \sqrt{r^2 - 4\left(\frac{n\pi c}{\ell}\right)^2}}{2} \equiv -\frac{r}{2} \pm i\frac{q_n}{2} ,$$

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where $q_n = \sqrt{\left(\frac{2n\pi c}{\ell}\right)^2 - r^2} > 0$ for $n = 1, 2, 3, \ldots$

The general solution to the D.E. (*) is

$$T_n(t) = e^{-\frac{rt}{2}}\left[a_n \cos\left(\frac{q_n t}{2}\right) + b_n \sin\left(\frac{q_n t}{2}\right)\right]$$

where a_n and b_n are arbitrary constants. Thus, the series expansion of the solution is

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t)$$

$$= \boxed{\sum_{n=1}^{\infty} e^{-\frac{rt}{2}} \sin\left(\frac{n\pi x}{\ell}\right)\left[a_n \cos\left(\frac{q_n t}{2}\right) + b_n \sin\left(\frac{q_n t}{2}\right)\right]}$$

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\#2. Consider a metal rod $(0 < x < \ell)$, insulated along its sides but not at its ends, which is initially at temperature $= 1$. Suddenly both ends are plunged into a bath of temperature $= 0$. Write the differential equation, boundary conditions, and initial condition. Write the formula for the temperature $u = u(x,t)$ at later times. In this problem, assume the infinite series expansion

$$(*) \qquad 1 = \frac{4}{\pi}\left(\sin\left(\frac{\pi x}{\ell}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{\ell}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{\ell}\right) + \cdots\right)$$

for $x \in (0, \ell)$.

The mathematical model for the problem is

$$\begin{cases} u_t - k u_{xx} = 0 & \text{for } 0 < x < \ell \text{ and } 0 < t < \infty, \\ u(0,t) = u(\ell,t) = 0 & \text{for } t \geq 0, \\ u(x,0) = 1 & \text{for } 0 < x < \ell. \end{cases}$$

By the work in the text, p. 85, the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{\ell}\right)^2 kt} \sin\left(\frac{n\pi x}{\ell}\right)$$

where the constants $A_1, A_2, A_3, \ldots$ are chosen so the initial condition is satisfied:

$$1 = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right) \quad \text{for } 0 < x < \ell.$$

From $(*)$ we see that $A_n = \begin{cases} 0 & \text{if } n = 2k \text{ is even}, \\ \frac{4}{\pi n} & \text{if } n = 2k+1 \text{ is odd}. \end{cases}$

Thus $$\boxed{u(x,t) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} e^{-\left[\frac{(2k+1)\pi}{\ell}\right]^2 kt} \sin\left(\frac{(2k+1)\pi x}{\ell}\right)}.$$

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#6. Separate variables for the equation

$$(*) \qquad t u_t = u_{xx} + 2u$$

with the boundary conditions $u(0,t) = u(\pi,t) = 0$. Show that there are an infinite number of solutions that satisfy the initial condition $u(x,0)=0$, so uniqueness fails for this equation.

Let $u(x,t) = X(x)T(t)$. Substituting this expression in $(*)$ and rearranging yields

$$-\frac{tT'}{T} + 2 = \frac{-X''}{X} = \text{constant} = \lambda .$$

Thus

$$(\dagger) \begin{cases} tT' - (\lambda+2)T = 0 \\ X'' + \lambda X = 0 , \qquad X(0) = X(\pi) = 0 . \end{cases}$$

The (nontrivial) solutions to the

$$X_n(x) = \sin(nx) , \quad (n = 1,2,3,\dots)$$

corresponding to the eigenvalues $\lambda_n = n^2$, $(n=1,2,3,\dots)$.
The solutions to the first O.D.E. in $(\dagger)$ are given by

$$\int \frac{T'(t)}{T(t)}\,dt = \int \frac{n^2+2}{t}\,dt$$

$$\ln(T(t)) = (n^2+2)\ln(t) + c = \ln\left(t^{n^2+2}\right) + c$$

$$\therefore \quad T_n(t) = A_n t^{2+n^2} \qquad (\text{where } A_n = e^c \text{ and } n=1,2,3,\dots).$$

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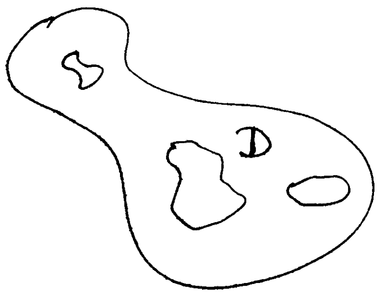
#6. (cont.) Therefore $u_n(x,t) = X_n(t)T_n(t) = A_n \sin(nx) t^{2+n^2}$, $(n=1,2,3,\dots)$ are solutions to the I-BVP

$$\begin{cases} t u_t = u_{xx} + 2u & \text{for } 0<x<\pi,\ 0<t<\infty, \\ u(0,t) = u(\pi,t) = 0 & \text{for } t \geq 0, \\ u(x,0) = 0 & \text{for } 0 \leq x \leq \pi. \end{cases}$$

Lecture Problem (A): Let $u = u(x,y)$ be a solution to $u_{xx} + u_{yy} = 0$ at all points (x,y) in an open bounded set D in the plane, and let u be continuous on $\overline{D} = D \cup \partial D$. Show that

$$\max_{(x,y)\in\partial D} u(x,y) = \max_{(x,y)\in\overline{D}} u(x,y) \quad \text{and} \quad \min_{(x,y)\in\partial D} u(x,y) = \min_{(x,y)\in\overline{D}} u(x,y).$$



Because D is bounded, there exists a number $L > 0$ such that $x^2+y^2 \leq L$ for all $(x,y) \in \partial D$. Let $\varepsilon > 0$ be given and set

$$v(x,y) = u(x,y) + \varepsilon(x^2+y^2)$$

for all points $(x,y) \in \overline{D}$. At any interior point $(x,y) \in D$,

$$(*) \qquad v_{xx} + v_{yy} = u_{xx} + 2\varepsilon + u_{yy} + 2\varepsilon = 4\varepsilon > 0.$$

If v has a maximum at an interior point $(x_0,y_0) \in D$ then $v_{xx}(x_0,y_0) \leq 0$ and $v_{yy}(x_0,y_0) \leq 0$. Therefore $v_{xx}(x_0,y_0) + v_{yy}(x_0,y_0) \leq 0$, in contradiction to $(*)$. Therefore v cannot attain a maximum value

Lecture Problem Ⓐ (cont.): at any point (x_0, y_0) in D. Consequently, the maximum value of the continuous function $v = v(x,y)$ on the closed bounded region $\overline{D}$ must occur at a point on ∂D. Hence

$$\begin{aligned}
\max_{(x,y)\in\overline{D}} u(x,y) &\leq \max_{(x,y)\in\overline{D}} \left\{ u(x,y) + \varepsilon(x^2+y^2) \right\} \\
&= \max_{(x,y)\in\overline{D}} v(x,y) \\
&= \max_{(x,y)\in\partial D} v(x,y) \\
&= \max_{(x,y)\in\partial D} \left\{ u(x,y) + \varepsilon(x^2+y^2) \right\} \\
&\leq \varepsilon L + \max_{(x,y)\in\partial D} u(x,y).
\end{aligned}$$

Since $\varepsilon > 0$ can be made arbitrarily small, it follows that

$$\max_{(x,y)\in\overline{D}} u(x,y) \leq \max_{(x,y)\in\partial D} u(x,y).$$

Because $\partial D \subset \overline{D}$, the reverse inequality clearly holds:

$$\max_{(x,y)\in\partial D} u(x,y) \leq \max_{(x,y)\in\overline{D}} u(x,y).$$

Consequently, $$\max_{(x,y)\in\partial D} u(x,y) = \max_{(x,y)\in\overline{D}} u(x,y).$$

In order to prove the analogous result for the minimum of u on ∂D and $\overline{D}$, respectively, consider the function defined by

Lecture Problem Ⓐ (cont.): $w(x,y) = -u(x,y)$ for $(x,y) \in \overline{D}$. Then w is a solution to $w_{xx} + w_{yy} = 0$ on D and is continuous on $\overline{D}$. By what has already been proved,

$$(**) \qquad \max_{(x,y)\in\partial D} w(x,y) = \max_{(x,y)\in\overline{D}} w(x,y).$$

But $\max_{(x,y)\in\partial D} w(x,y) = \max_{(x,y)\in\partial D} -u(x,y) = -\min_{(x,y)\in\partial D} u(x,y)$ and similarly

$\max_{(x,y)\in\overline{D}} w(x,y) = -\min_{(x,y)\in\overline{D}} u(x,y)$. Substituting these expressions in $(**)$ and simplifying yields the desired result:

$$\min_{(x,y)\in\partial D} u(x,y) = \min_{(x,y)\in\overline{D}} u(x,y).$$

Lecture Problem Ⓑ. Let D be an open bounded subset of the plane, let f be a given function on D, and let φ be a given function on ∂D. Then there is at most one function $u = u(x,y)$ which is continuous on $\overline{D}$ and which solves the B.V.P.

$$(\dagger) \quad \begin{cases} u_{xx} + u_{yy} = f & \text{on } D, \\ u = \varphi & \text{on } \partial D. \end{cases}$$

Proof: Suppose that $u = u_1(x,y)$ and $u = u_2(x,y)$ are continuous on $\overline{D}$ and solve $(\dagger)$. Then $u = u_1(x,y) - u_2(x,y)$ is continuous on $\overline{D}$ and solves

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{on } D, \\ u = 0 & \text{on } \partial D. \end{cases}$$

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Lecture Problem (B) (cont.): By the maximum/minimum principle for Laplace's equation (Lecture Problem (A))

$$0 = \max_{(x,y)\in\overline{D}} u(x,y) \quad \text{and} \quad 0 = \min_{(x,y)\in\overline{D}} u(x,y)$$

Therefore $u(x,y) = 0$ for all $(x,y) \in \overline{D}$; i.e. $u_1(x,y) = u_2(x,y)$ for all $(x,y) \in \overline{D}$.