Sec. 4.2, p.90.

#1. Solve the diffusion equation $u_t - ku_{xx} = 0$ for $0 < x < l$ and $0 < t < \infty$ with the mixed boundary conditions $u(0,t) = u_x(l,t) = 0$ for $t \geq 0$.

We seek nontrivial solutions via the method of separation of variables. Let $u(x,t) = X(x)T(t)$. Then substituting in the PDE produces $XT' - kX''T = 0$ or

$$\frac{T'}{kT} = \frac{X''}{X} = \text{constant} = \lambda .$$

Hence
$$\begin{cases} T' - \lambda k T = 0, \\ X'' - \lambda X = 0, \quad X(0) = 0 = X'(l). \end{cases}$$

Case $\lambda > 0$, say $\lambda = \beta^2$: The solution to the ODE is $X(x) = c_1\cosh(\beta x) + c_2\sinh(\beta x)$. The B.C.'s yield:

$$\left.\begin{aligned} 0 = X(0) &= c_1 \cdot 1 + c_2 \cdot 0 \Rightarrow c_1 = 0 \\ 0 = X'(l) &= \beta \cancel{c_1}^{0}\sinh(\beta l) + \beta c_2\cosh(\beta l) \Rightarrow c_2 = 0 \end{aligned}\right\} \text{Trivial solution.}$$

Case $\lambda = 0$: The solution to the ODE is $X(x) = c_1 x + c_2$. The B.C.'s yield

$$\left.\begin{aligned} 0 = X(0) &= c_1 \cdot 0 + c_2 \Rightarrow c_2 = 0 \\ 0 = X'(l) &= c_1 \end{aligned}\right\} \text{Trivial solution.}$$

Case $\lambda < 0$, say $\lambda = -\beta^2$: The solution to the ODE is $X(x) = c_1\cos(\beta x) + c_2\sin(\beta x)$. The B.C.'s yield

$$0 = X(0) = c_1 \cdot 1 + c_2 \cdot 0 \Rightarrow c_1 = 0$$

$$0 = X'(l) = -\beta \cancel{c_1}^{0}\sin(\beta l) + \beta c_2\cos(\beta l) \Rightarrow \beta = \beta_n = (2n+1)\frac{\pi}{2l} = \left(n+\tfrac{1}{2}\right)\frac{\pi}{l}$$
$$(n = 0, 1, 2, \ldots)$$

Thus $\lambda_n = -\left(n+\frac{1}{2}\right)^2\frac{\pi^2}{l^2}$ $(n = 0, 1, 2, 3, \ldots)$ are the eigenvalues and $X_n(x) = c_n \sin\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right)$ $(n = 0, 1, 2, \ldots)$ are the corresponding eigenfunctions.

Sec. 4.2, p. 90.

#2. (cont.)

(a) Let $u(x,t) = X(x)T(t)$. Then $u_{tt} = X(x)T''(t)$ and $u_{xx} = X''(x)T(t)$ so the PDE becomes $T''X = c^2 X''T \Rightarrow \frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda$.

Therefore $\begin{cases} T'' + \lambda c^2 T = 0 \\ \boxed{X'' + \lambda X = 0} \end{cases}$

The B.C.'s are $X'(0)T(t) = 0 = X(l)T(t)$ for all $t \geq 0$. Since $T(t) \not\equiv 0$, it must happen that $\boxed{X(l) = X'(0) = 0}$.

Case $\lambda > 0$ (say $\boxed{\lambda = \beta^2} > 0$): $X(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x)$ is the general solution to the ODE. $X'(x) = -\beta c_1 \sin(\beta x) + \beta c_2 \cos(\beta x)$.

$\therefore$ $X'(0) = 0 \Rightarrow \beta c_2 = 0 \Rightarrow c_2 = 0$. Thus $X(x) = c_1 \cos(\beta x)$.

$X(l) = 0 \Rightarrow c_1 \cos(\beta l) = 0 \Rightarrow \cos(\beta l) = 0 \Rightarrow \beta_n l = (n + \frac{1}{2})\pi$

$\Rightarrow \boxed{\beta_n = \frac{(n+\frac{1}{2})\pi}{l}}$ (for some integer $n = 0, 1, 2, 3, \ldots$). The eigenfunctions are

$X_n(x) = \cos(\beta_n x) = \boxed{\cos\left(\frac{(n+\frac{1}{2})\pi x}{l}\right)}$, $n = 0, 1, 2, \ldots$

Case $\lambda = 0$: $X(x) = c_1 x + c_2$ is the general solution to the ODE.

$0 = X'(0) = c_1 \Rightarrow X(x) = c_2$. $0 = X(l) = c_2 \Rightarrow X(x) \equiv 0$.

that is, 0 is not an eigenvalue.

Case $\lambda < 0$ (say $\lambda = -\beta^2 < 0$): $X(x) = c_1 \cosh(\beta x) + c_2 \sinh(\beta x)$ is the general solution to the ODE. $X'(x) = \beta c_1 \sinh(\beta x) + \beta c_2 \cosh(\beta x)$.

$\therefore$ $0 = X'(0) = \beta c_2 \Rightarrow c_2 = 0$. Therefore $X(x) = c_1 \cosh(\beta x)$.

$0 = X(l) = c_1 \cosh(\beta l) \Rightarrow c_1 = 0$ because $\cosh(t) \neq 0$ for all real t.

Therefore $X(x) \equiv 0$; that is, there are no eigenvalues in this case.

Sec. 4.2, #2, p.90 (cont.)

(b) The eigenvalues are $\lambda_n = \beta_n^2 = (n+\frac{1}{2})^2\pi^2/\ell^2$ $(n = 0, 1, 2, \ldots)$, so the ODE for T is $T'' + \beta_n^2 c^2 T = 0$. The general solution is $T_n(t) = a_n \cos(\beta_n ct) + b_n \sin(\beta_n ct)$. Thus, the formal series expansion for a solution $u = u(x,t)$ is

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} \underbrace{\left[a_n \cos\left((n+\tfrac{1}{2})\pi ct/\ell\right) + b_n \sin\left((n+\tfrac{1}{2})\pi ct/\ell\right)\right]}_{T_n(t)} \underbrace{\cos\left(\frac{(n+\frac{1}{2})\pi x}{\ell}\right)}_{X_n(x)}}$$