Sec. 5.1 The (Fourier) Coefficients, pp. 108–109.

#2, p. 108. Let $\varphi(x) = x^2$ for $0 \le x \le 1$.

(a) Calculate its Fourier sine series.

(b) Calculate its Fourier cosine series.

(a) $$b_n = \frac{\langle \varphi, \sin(n\pi x)\rangle}{\langle \sin(n\pi x), \sin(n\pi x)\rangle} = \frac{\int_0^1 \varphi(x)\sin(n\pi x)\,dx}{\int_0^1 \sin^2(n\pi x)\,dx}.$$

$$\int_0^1 \sin^2(n\pi x)\,dx = \int_0^1 \frac{1-\cos(2n\pi x)}{2}\,dx = \frac{1}{2}x - \frac{1}{4n\pi}\sin(2n\pi x)\Big|_0^1 = \frac{1}{2}.$$

$$\therefore\ b_n = 2\int_0^1 x^2 \sin(n\pi x)\,dx$$

Parts: $U = x^2$, $dV = \sin(n\pi x)dx$
$\Rightarrow$ $dU = 2x\,dx$, $V = -\frac{\cos(n\pi x)}{n\pi}$

$$= 2\left(-\frac{x^2\cos(n\pi x)}{n\pi}\right)\Bigg|_0^1 + \frac{4}{\pi n}\int_0^1 x\cos(n\pi x)\,dx$$

Parts: $U = x$, $dV = \cos(n\pi x)dx$
$\Rightarrow$ $dU = dx$, $V = \frac{\sin(n\pi x)}{n\pi}$

$$= \frac{2(-1)^n}{n\pi} + \frac{4}{n\pi}\left(\frac{x\sin(n\pi x)}{n\pi}\right)\Bigg|_0^1 - \frac{4}{(n\pi)^2}\int_0^1 \sin(n\pi x)\,dx$$

$$= \frac{2(-1)^n}{n\pi} + \frac{4}{(n\pi)^3}\cos(n\pi x)\Big|_0^1$$

$$= \frac{2(-1)^n}{n\pi} + \frac{4}{(n\pi)^3}\left[(-1)^n - 1\right]$$

this factor is 0 if n is even and −2 if n is odd.

Fourier sine series of φ:
$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\sin(n\pi x)}{n} - \frac{8}{\pi^3}\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi x)}{(2k+1)^3}$$

Sec. 5.1

#2, p.108 (b)

$$a_n = \frac{\langle \varphi, \cos(n\pi x)\rangle}{\langle \cos(n\pi x), \cos(n\pi x)\rangle} = \frac{\int_0^1 \varphi(x)\cos(n\pi x)\,dx}{\int_0^1 \cos^2(n\pi x)\,dx}.$$

$$(n \geq 1) \quad \int_0^1 \cos^2(n\pi x)\,dx = \int_0^1 \frac{1+\cos(2n\pi x)}{2}\,dx = \frac{1}{2}x + \frac{\sin(2n\pi x)}{4n\pi}\Big|_0^1 = \frac{1}{2}.$$

$$\therefore \quad a_n = 2\int_0^1 x^2\cos(n\pi x)\,dx$$

Parts: $U = x^2$, $dV = \cos(n\pi x)dx$
$\Rightarrow$ $dU = 2x\,dx$, $V = \frac{\sin(n\pi x)}{n\pi}$

$$= \cancel{\frac{2x^2\sin(n\pi x)}{n\pi}\Big|_0^1}^{\;0} - \frac{4}{n\pi}\int_0^1 x\sin(n\pi x)\,dx$$

Parts: $U = x$, $dV = \sin(n\pi x)dx$
$\Rightarrow$ $dU = dx$, $V = \frac{-\cos(n\pi x)}{n\pi}$

$$= \frac{4x\cos(n\pi x)}{(n\pi)^2}\Big|_0^1 - \cancel{\frac{4}{(n\pi)^2}\int_0^1 \cos(n\pi x)\,dx}^{\;0}$$

$$= \frac{4(-1)^n}{(n\pi)^2} \quad (n \geq 1)$$

$$a_0 = \frac{\langle \varphi, 1\rangle}{\langle 1, 1\rangle} = \frac{\int_0^1 \varphi(x)\,dx}{\int_0^1 1^2\,dx} = \int_0^1 x^2\,dx = \frac{1}{3}.$$

Fourier cosine series of φ $\Big\}$

$$\frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n\cos(n\pi x)}{n^2}$$

Sec. 5.1

#3, p. 108. Consider the function $\varphi(x) = x$ on $(0, \ell)$. On the same graph, sketch the following functions.

(a) The sum of the first three (nonzero) terms of its Fourier sine series.

(b) The sum of the first three (nonzero) terms of its Fourier cosine series.

In examples 3 and 4 of Sec. 5.1 it is shown that the Fourier sine and cosine series of φ are, respectively,

$$\frac{2\ell}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \sin(m\pi x/\ell)}{m}$$

and

$$\frac{\ell}{2} - \frac{4\ell}{\pi^2} \sum_{k=0}^{\infty} \frac{\cos((2k+1)\pi x/\ell)}{(2k+1)^2} .$$

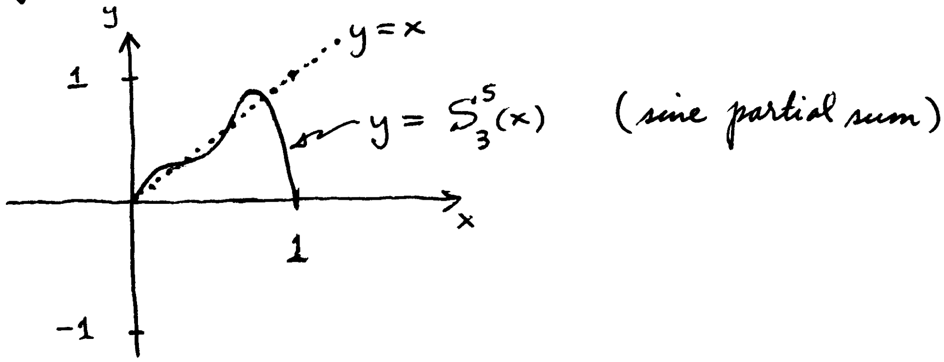
The first three (nonzero) terms of the respective series are

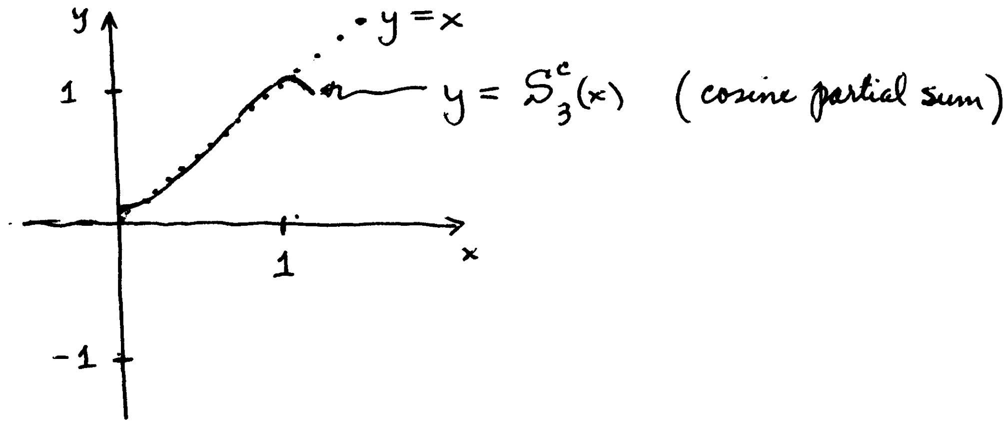
$$S_3^s(x) = \frac{2\ell}{\pi}\left(\sin(\pi x/\ell) - \frac{1}{2}\sin(2\pi x/\ell) + \frac{1}{3}\sin(3\pi x/\ell) \right)$$

and

$$S_3^c(x) = \frac{\ell}{2} - \frac{4\ell}{\pi^2}\left(\cos(\pi x/\ell) + \frac{1}{9}\cos(3\pi x/\ell) \right).$$

#3, p.108 (cont.) Using a graphical calculator we find the following graphs when $l = 1$.





#5, p.108. Consider the Fourier sine series of $\varphi(x) = x$ on $(0, l)$:

$$\frac{2l}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \sin(m\pi x/l)}{m} = \frac{2l}{\pi}\left(\sin\left(\frac{\pi x}{l}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{l}\right) + \dots \right).$$

(See example 3 of Sec. 5.1 for details of this computation.) Assume that the series can be integrated term-by-term, a fact that will be shown later (cf. Theorem 2 of Sec. 5.4).

(a) Find the Fourier cosine series of the function

#5, p.108 (cont.) $x^2/2$. Find the constant of integration that will be the first term in the cosine series.

(b) Then by setting $x=0$ in your result, find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

We also assume that $\varphi(x)=x$ on $(0,\ell)$ is equal to its Fourier sine series there:

$$(*) \qquad t = \frac{2\ell}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}\sin(m\pi t/\ell)}{m} \quad \text{if } 0 \leq t < \ell.$$

(See Theorem 4(i) of Sec. 5.4.) If $x \in [0,\ell)$ then, by integrating $(*)$ over $[0,x]$, we have

$$\frac{x^2}{2} = \int_0^x t\,dt = \int_0^x \left(\frac{2\ell}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}\sin(m\pi t/\ell)}{m}\right)dt$$

$$= \frac{2\ell}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}\int_0^x \sin(m\pi t/\ell)\,dt$$

$$= \frac{2\ell}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}\left[-\frac{\ell}{m\pi}\cos\left(\frac{m\pi t}{\ell}\right)\right]\Bigg|_{t=0}^{t=x}$$

$$= \frac{2\ell^2}{\pi^2}\sum_{m=1}^{\infty} \frac{(-1)^{m}\cos(m\pi x/\ell)}{m^2} + c$$

#5, p.108 (cont.) where $c = \frac{2\ell^2}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^2}$.

But c must also be the zeroth Fourier cosine coefficient of $x^2/2$ on $(0, \ell)$. Thus

$$c = \frac{\langle x^2/2, 1 \rangle}{\langle 1, 1 \rangle} = \frac{\int_0^\ell x^2/2 \, dx}{\int_0^\ell 1^2 \, dx} = \frac{\ell^3/6}{\ell}$$

$$c = \ell^2/6 .$$

Equating the two expressions for c gives

$$\frac{\ell^2}{6} = c = \frac{2\ell^2}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^2} ;$$

i.e. $$\boxed{\frac{\pi^2}{12}} = \frac{\ell^2}{6} \cdot \frac{\pi^2}{2\ell^2} = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^2} .$$

Note also that the expression $c = \ell^2/6$ can be substituted in the formula for $x^2/2$ on $(0, \ell)$ at the bottom of the previous page to yield its Fourier cosine series:

$$\boxed{\frac{x^2}{2} = \frac{\ell^2}{6} + \frac{2\ell^2}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^m \cos(m\pi x/\ell)}{m^2}}, \quad x \in (0, \ell) .$$

Sec. 5.1

#6, p. 108 (a) By the same method (as #5), find the sine series of x^3.
(b) Find the cosine series of x^4.

Using the result of #5,

$$\frac{x^2}{2} = \frac{\ell^2}{6} + \frac{2\ell^2}{\pi^2}\sum_{m=1}^{\infty} \frac{(-1)^m \cos(m\pi x/\ell)}{m^2}, \qquad x \in [0, \ell),$$

and integrating we find

$$(*) \qquad \frac{x^3}{6} = \frac{\ell^2 x}{6} + \frac{2\ell^2}{\pi^2}\sum_{m=1}^{\infty} \frac{\ell\,(-1)^m \sin(m\pi x/\ell)}{m\pi \cdot m^2} + c$$

for $x \in [0, \ell)$. Substituting the previous result (Example 3, Sec. 5.1)

$$x = \frac{2\ell}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^{m+1} \sin(m\pi x/\ell)}{m}, \qquad x \in [0, \ell),$$

into (*) and simplifying yields

$$\frac{x^3}{6} = \frac{\ell^3}{3\pi}\sum_{m=1}^{\infty} \frac{(-1)^{m+1} \sin(m\pi x/\ell)}{m} + \frac{2\ell^3}{\pi^3}\sum_{m=1}^{\infty} \frac{(-1)^m \sin(m\pi x/\ell)}{m^3} + c$$

$$\boxed{x^3 = \frac{2\ell^3}{\pi^3}\sum_{m=1}^{\infty} (-1)^m \left(\frac{6}{m^3} - \frac{\pi^2}{m}\right) \sin(m\pi x/\ell)} + \cancel{c}^{\,0}.$$

Setting $x = 0$, we find that $c = 0$ above.

#6, p.108 (b) (cont.)

Integrating term-by-term the Fourier sine series for x^3 yields

$$\frac{x^4}{4} = \frac{2\ell^3}{\pi^3} \sum_{m=1}^{\infty} (-1)^m \left(\frac{6}{m^3} - \frac{\pi^2}{m} \right) \left[-\cos(m\pi x/\ell) \right] \frac{\ell}{m\pi} + c$$

$$(**) \qquad x^4 = \frac{8\ell^4}{\pi^4} \sum_{m=1}^{\infty} (-1)^{m+1} \left(\frac{6}{m^4} - \frac{\pi^2}{m^2} \right) \cos(m\pi x/\ell) + c'$$

for $x \in [0, \ell)$. Setting $x = 0$ gives

$$\frac{8\ell^4}{\pi^4} \sum_{m=1}^{\infty} (-1)^m \left(\frac{6}{m^4} - \frac{\pi^2}{m^2} \right) = c'.$$

Also, c' must be the zeroth Fourier cosine coefficient of x^4 on $[0, \ell)$:

$$c' = \frac{\langle x^4, 1 \rangle}{\langle 1, 1 \rangle} = \frac{\int_0^\ell x^4 \, dx}{\int_0^\ell 1^2 \, dx} = \frac{\ell^5/5}{\ell} = \frac{\ell^4}{5}.$$

Equating the two expressions for c' yields

$$(\dagger) \qquad \frac{48 \cancel{\ell^4}}{\pi^4} \sum_{m=1}^{\infty} \frac{(-1)^m}{m^4} + \frac{8 \cancel{\ell^4}}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^2} = \frac{\cancel{\ell^4}}{5}.$$

Substituting the result from #5(b) into ($\dagger$) gives

$$\frac{48}{\pi^4} \cdot \sum_{m=1}^{\infty} \frac{(-1)^m}{m^4} + \frac{8}{\pi^2} \cdot \frac{\pi^2}{12} = \frac{1}{5},$$

#6, p.108 (b) (cont.) and simplifying yields

$$\boxed{\sum_{m=1}^{\infty} \frac{(-1)^m}{m^4} = \frac{\pi^4}{48}\left(\frac{1}{5}-\frac{2}{3}\right) = \frac{-7\pi^4}{720}}$$

This is the answer to #7.

Substituting the expression $c' = \ell^4/5$ into (**) produces

$$x^4 = \frac{\ell^4}{5} + \frac{8\ell^4}{\pi^4}\sum_{m=1}^{\infty}(-1)^{m+1}\left(\frac{6}{m^4}-\frac{\pi^2}{m^2}\right)\cos(m\pi x/\ell)$$

for $x \in [0,\ell)$.

#7, p.108. Put $x=0$ in exercise #6(b) to deduce the sum of the series

$$\sum_{m=1}^{\infty}\frac{(-1)^m}{m^4}.$$

(Solution above.)

Sec. 5.1

#8, p. 108. A rod has length $l = 1$ and constant $k = 1$. Its temperature satisfies the heat equation. Its left end is held at temperature 0, its right end at temperature 1. Initially (at $t = 0$) the temperature is given by

$$\varphi(x) = \begin{cases} 5x/2 & \text{if } 0 < x < 2/3, \\ 3 - 2x & \text{if } 2/3 < x < 1. \end{cases}$$

Find the solution (i.e. the temperature as a function of position x and time t), including the coefficients.

Mathematical Model:

$$(*) \qquad u_t - u_{xx} = 0 \quad \text{for } 0 < x < 1,\ 0 < t < \infty,$$

$$(**) \qquad u(0,t) = 0 \text{ and } u(1,t) = 1 \quad \text{for } t \geq 0$$

$$(***) \qquad u(x,0) = \varphi(x) = \begin{cases} 5x/2 & \text{if } 0 < x < 2/3, \\ 3 - 2x & \text{if } 2/3 < x < 1. \end{cases}$$

Note: Direct application of the method of separation of variables fails because the nonhomogeneous boundary condition $u(1,t) = 1$ does not lead to an eigenvalue problem. (Try it, you'll see what I mean.) Therefore, we seek the steady-state solution

$$u(x,t) = U(x) \qquad \text{(independent of time } t\text{)}$$

to the above problem (*)-(**). The substitution $v(x,t) = u(x,t) - U(x)$ in (*)-(**)-(***) will then lead to a problem with homogeneous boundary conditions, and hence, one solvable by separation of variables.

#8, p. 108 (cont.) The steady-state solution $u(x,t) = U(x)$ to (*)-(**) solves

$$0 - U'' = 0 \quad \text{for} \quad 0<x<1 \qquad (*')$$

$$U(0) = 0 \quad \text{and} \quad U(1) = 1. \qquad (**')$$

The general solution to the ODE (*') is $U(x) = c_1 x + c_2$. The B.C.'s (**') imply $c_2 = 0$ and $c_1 = 1$. Thus $U(x) = x$ is the steady-state solution to (*)-(**).

Make the substitution $v(x,t) = u(x,t) - U(x) = u(x,t) - x$ in the problem (*)-(**)-(***). This results in the problem

$$(\dagger)\begin{cases} v_t - v_{xx} = 0 \quad \text{for } 0<x<1,\ 0<t<\infty, \\ \left.\begin{array}{l} v(0,t) = \overbrace{u(0,t)}^{0} - \overbrace{U(0)}^{0} = 0 \\ \text{and} \\ v(1,t) = \underbrace{u(1,t)}_{1} - \underbrace{U(1)}_{1} = 0 \end{array}\right\} \text{for } t \geq 0, \\ v(x,0) = u(x,0) - U(x) = \overbrace{\varphi(x) - x}^{\text{Call this } \psi(x)} = \begin{cases} \frac{3x}{2} & \text{if } 0<x<2/3, \\ 3-3x & \text{if } 2/3<x<1. \end{cases} \end{cases}$$

We use separation of variables to solve ($\dagger$). The form $v(x,t) = X(x)T(t)$ leads to

$$T'(t) + \lambda T(t) = 0,$$

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(1) = 0.$$

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#8, p.108 (cont.) The eigenvalues are $\lambda_n = (n\pi)^2$, $n = 1, 2, 3, \dots$, and the corresponding eigenfunctions are $X_n(x) = \sin(n\pi x)$, $n = 1, 2, 3, \dots$. Therefore

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-(n\pi)^2 t} \qquad (b_1, b_2, \dots \text{ arbitrary consts.})$$

is a formal solution to the homogeneous part of (†). To satisfy the initial condition $v(x,0) = \psi(x) = \begin{cases} 3x/2 & \text{if } 0 < x < 2/3, \\ 3 - 3x & \text{if } 2/3 < x < 1, \end{cases}$

we need to choose the constants $b_1, b_2, \dots$ to be the Fourier sine coefficients of ψ:

$$b_n = \frac{\langle \psi, \sin(n\pi \cdot) \rangle}{\langle \sin(n\pi \cdot), \sin(n\pi \cdot) \rangle} = \frac{\int_0^1 \psi(x) \sin(n\pi x)\,dx}{\int_0^1 \sin^2(n\pi x)\,dx} \leftarrow \tfrac{1}{2}$$

$$b_n = 2\int_0^1 \psi(x) \sin(n\pi x)\,dx = \cancel{2}\int_0^{2/3} \overbrace{\frac{3x}{\cancel{2}}}^{u} \overbrace{\sin(n\pi x)\,dx}^{dv} + 2\int_{2/3}^1 \overbrace{(3-3x)}^{u} \overbrace{\sin(n\pi x)\,dx}^{dv}.$$

Integrating by parts (as indicated), we find

$$b_n = \left.\frac{-3x\cos(n\pi x)}{n\pi}\right|_0^{2/3} + \frac{3}{n\pi}\int_0^{2/3} \cos(n\pi x)\,dx + \left.\frac{-6(1-x)\cos(n\pi x)}{n\pi}\right|_{2/3}^1 - \frac{6}{n\pi}\int_{2/3}^1 \cos(n\pi x)\,dx$$

$$= \cancel{-\frac{2\cos(2n\pi/3)}{n\pi}} + \left.\frac{3\sin(n\pi x)}{(n\pi)^2}\right|_0^{2/3} + \cancel{\frac{2\cos(2n\pi/3)}{n\pi}} - \left.\frac{6\sin(n\pi x)}{(n\pi)^2}\right|_{2/3}^1$$

$$= \frac{9\sin(2n\pi/3)}{(n\pi)^2}.$$

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#8, p.108 (cont.) Therefore

$$v(x,t) = \frac{9}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin(2n\pi/3)\sin(n\pi x)e^{-(n\pi)^2 t}}{n^2},$$

and hence

$$u(x,t) = v(x,t) + x$$

$$= \boxed{x + \frac{9}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin(2n\pi/3)\sin(n\pi x)e^{-(n\pi)^2 t}}{n^2}}.$$

#10, p.108. A string (of tension T and density ρ), with fixed ends at $x=0$ and $x=\ell$, is hit by a hammer so that $u(x,0)=0$ for $x \in [0,\ell]$ and

$$u_t(x,0) = \begin{cases} V & \text{if } |x-\frac{\ell}{2}| \le \delta, \\ 0 & \text{o.w.} \end{cases}$$

Find the solution explicitly in series form. Find the energy

$$E_n(h) = \frac{1}{2}\int_0^{\ell} \left[\rho u_t^2(x,t) + T u_x^2(x,t)\right] dx$$

of the n^{th} harmonic $h = h_n$. Conclude that if δ is small (a

#10, p. 108 (cont.) concentrated blow), each of the first few overtones has almost as much energy as the fundamental. We could say that the tone is saturated with overtones.

Mathematical Model:

$$(*) \quad \rho u_{tt} - T u_{xx} = 0 \quad \text{for } 0 < x < \ell,\ 0 < t < \infty,$$

$$(**) \quad u(0,t) = u(\ell,t) = 0 \quad \text{for } t \geq 0,$$

$$(***) \quad u(x,0) = 0 \quad \text{for } 0 \leq x \leq \ell,$$

$$(****) \quad u_t(x,0) = \psi(x) = \begin{cases} V & \text{if } \left|x - \frac{\ell}{2}\right| \leq \delta \\ 0 & \text{if } \delta < \left|x - \frac{\ell}{2}\right| \leq \frac{\ell}{2}. \end{cases}$$

Separation of variables (i.e. $u(x,t) = X(x)T(t)$) leads to the formal solution

$$(\dagger) \quad u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{n\pi c t}{\ell}\right)$$

to the homogeneous part of the problem: (*)-(**)-(***). Here $c = \sqrt{T/\rho}$ and $c_1, c_2, \ldots$ are arbitrary constants. Differentiating the formal solution with respect to t gives (formally)

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi c c_n}{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{n\pi c t}{\ell}\right),$$

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#10, p.108 (cont.) so if (****) is to be satisfied, we must have

$$\frac{n\pi c\, c_n}{\ell} = n^{th} \text{ Fourier sine coefficient of } \psi$$

$$= \frac{\langle \psi, \sin(n\pi(\cdot)/\ell)\rangle}{\langle \sin(n\pi(\cdot)/\ell), \sin(n\pi(\cdot)/\ell)\rangle}$$

$$= \frac{\int_0^\ell \psi(x)\sin(n\pi x/\ell)\,dx}{\int_0^\ell \sin^2(n\pi x/\ell)\,dx} \longleftarrow \ell/2$$

$$= \frac{2}{\ell}\int_{\frac{\ell}{2}-\delta}^{\frac{\ell}{2}+\delta} V \sin(n\pi x/\ell)\,dx$$

$$= \frac{2V}{\ell}\left[-\frac{\ell}{n\pi}\cos\left(\frac{n\pi x}{\ell}\right)\right]\Bigg|_{\frac{\ell}{2}-\delta}^{\frac{\ell}{2}+\delta}$$

$$= \frac{2V}{n\pi}\left[\cos\left[\frac{n\pi}{\ell}\left(\frac{\ell}{2}-\delta\right)\right] - \cos\left[\frac{n\pi}{\ell}\left(\frac{\ell}{2}+\delta\right)\right]\right].$$

We now apply the identity $\cos(B) - \cos(A) = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

with $B = \frac{n\pi}{\ell}\left(\frac{\ell}{2}-\delta\right) = \frac{n\pi}{2} - \frac{n\pi\delta}{\ell}$

and $A = \frac{n\pi}{\ell}\left(\frac{\ell}{2}+\delta\right) = \frac{n\pi}{2} + \frac{n\pi\delta}{\ell}$.

Note that $\frac{A+B}{2} = \frac{n\pi}{2}$ and $\frac{A-B}{2} = \frac{n\pi\delta}{\ell}$. Therefore

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#10, p. 108 (cont.) $$\frac{n\pi c c_n}{\ell} = \frac{4V}{n\pi}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi\delta}{\ell}\right) \qquad (n=1,2,3,\ldots).$$

But $\sin\left(\frac{n\pi}{2}\right) = 0$ if $n=2m$ is even and $\sin\left(\frac{n\pi}{2}\right) = (-1)^{m-1}$ if $n=2m-1$ is odd. Thus

$$(\dagger\dagger) \qquad c_n = \frac{4\ell V}{c(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi\delta}{\ell}\right) = \begin{cases} 0 & \text{if } n=2m \text{ is even,} \\ \dfrac{4\ell V(-1)^{m-1}\sin((2m-1)\pi\delta/\ell)}{c\pi^2(2m-1)^2} & \text{if } n=2m-1 \text{ is odd.} \end{cases}$$

Consequently, substituting $(\dagger\dagger)$ into $(\dagger)$ gives

$$\boxed{u(x,t) = \frac{4\ell V}{c\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}\sin((2m-1)\pi\delta/\ell)\sin((2m-1)\pi x/\ell)\sin((2m-1)\pi c t/\ell)}{(2m-1)^2}}$$

where $c = \sqrt{T/\rho}$.

The n^{th} harmonic of the series solution $(\dagger)$ is the n^{th} term:

$$h_n(x,t) = c_n \sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{n\pi c t}{\ell}\right)$$

where c_n is given by $(\dagger\dagger)$. The energy of the n^{th} harmonic is

$$E(h_n)(t) = \frac{1}{2}\int_0^{\ell}\left[\rho\left(\frac{\partial h_n}{\partial t}\right)^2 + T\left(\frac{\partial h_n}{\partial x}\right)^2\right]dx.$$

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#10, p.108 (cont.) Since

$$\frac{\partial h_n}{\partial t} = \frac{n\pi c c_n}{l} \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi c t}{l}\right)$$

and

$$\frac{\partial h_n}{\partial x} = \frac{n\pi c_n}{l} \cos\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi c t}{l}\right),$$

upon substituting into the integral expression for $E(h_n)(t)$, we get

$$E(h_n)(t) = \frac{1}{2}\rho c^2 \left(\frac{n\pi c_n}{l}\right)^2 \cos^2\left(\frac{n\pi c t}{l}\right) \int_0^l \sin^2\left(\frac{n\pi x}{l}\right) dx$$
$$+ \frac{1}{2} T \left(\frac{n\pi c_n}{l}\right)^2 \sin^2\left(\frac{n\pi c t}{l}\right) \int_0^l \cos^2\left(\frac{n\pi x}{l}\right) dx.$$

But $\rho c^2 = \cancel{\rho} \cdot T/\cancel{\rho} = T$,

$$\int_0^l \sin^2\left(\frac{n\pi x}{l}\right) dx = \int_0^l \left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2n\pi x}{l}\right)\right] dx = \frac{l}{2},$$

and $$\int_0^l \cos^2\left(\frac{n\pi x}{l}\right) dx = \int_0^l \left[\frac{1}{2} + \frac{1}{2}\cos\left(\frac{2n\pi x}{l}\right)\right] dx = \frac{l}{2}.$$

Thus $$E(h_n)(t) = \frac{Tl}{4}\left(\frac{n\pi c_n}{l}\right)^2 \cos^2\left(\frac{n\pi c t}{l}\right) + \frac{Tl}{4}\left(\frac{n\pi c_n}{l}\right)^2 \sin^2\left(\frac{n\pi c t}{l}\right)$$

$$= \frac{T n^2 \pi^2 c_n^2}{4l}. \quad \leftarrow \left(\text{constant, independent of } t!\right)$$

#10, p. 108 (cont.) Substituting the expression for c_n from (††) yields

$$E(h_n) = \frac{T n^2\pi^2}{4\ell}\left(\frac{4\ell V}{c(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi\delta}{\ell}\right)\right)^2$$

$$= \frac{4\rho\ell V^2}{(n\pi)^2}\sin^2\left(\frac{n\pi}{2}\right)\sin^2\left(\frac{n\pi\delta}{\ell}\right)$$

$$= \begin{cases} 0 & \text{if } n=2m \text{ is even,} \\ \dfrac{4\rho\ell V^2}{(n\pi)^2}\sin^2\left(\dfrac{n\pi\delta}{\ell}\right) & \text{if } n=2m-1 \text{ is odd.} \end{cases}$$

Suppose δ is "small", corresponding physically to a localized blow. Using the approximation $\sin\theta \approx \theta$ when θ is "small", we see that for the first few odd values of n that

$$E(h_n) \approx \frac{4\rho\ell V^2}{(n\pi)^2}\cdot\left(\frac{n\pi\delta}{\ell}\right)^2 = \frac{4\rho V^2\delta^2}{\ell}$$

<u>independent</u> of n! Thus the first few $\overbrace{\text{overtones}}^{h_n}$ have almost the same energy as the fundamental $h_1 = h_1(x,t)$.

Sec. 5.1

#11, p.109. On a string with fixed ends, show that if the center of a hammer blow is exactly at a node of the n^{th} harmonic (i.e. at a place where the n^{th} eigenfunction vanishes), the n^{th} overtone is absent from the solution.

The mathematical model for the problem will be (*)-(**)-(***) as in #10, p.108, but the equation modeling the blow (****) will need to be modified. Note that the N^{th} eigenfunction is

$$X_N(x) = \sin\left(\frac{N\pi x}{\ell}\right)$$

where $N \geq 1$ is an integer. The N^{th} eigenfunction has nodes in the interior of the interval $(0,\ell)$ at

$$x_k = \frac{k\ell}{N} \quad \text{where} \quad k = 1, 2, \ldots, N-1.$$

Suppose the blow were centered exactly at one of these nodes, say at $x_K = \frac{K\ell}{N}$ where $1 \leq K \leq N-1$. Then the mathematical model for the problem is (*)-(**)-(***) from #10, p.108, plus

$$(****') \qquad u_t(x,0) = \psi(x) = \begin{cases} V & \text{if } \left|x - \frac{K\ell}{N}\right| \leq \delta, \\ 0 & \text{otherwise.} \end{cases}$$

As in #10, p.108, the series solution is

Sec. 5.1

#11, p.109 (cont.)

$$(\dagger) \qquad u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi c t}{l}\right)$$

where $\frac{n\pi c c_n}{l}$ = n^{th} Fourier sine coefficient of ψ $(n = 1, 2, 3, \dots)$.

To show that the N^{th} overtone is absent from the solution, we must show that $c_N = 0$. In fact, we will show more, for we shall show that $0 = c_N = c_{2N} = c_{3N} = \dots$

Let $j \geq 1$ be an integer. Using the formula above for c_n with $n = jN$, and proceeding as in #10, p.108 we find

$$\frac{jN\pi c c_{jN}}{l} = (jN)^{th} \text{ Fourier sine coefficient of } \psi \text{ (given in (} \cancel{****} \text{))}$$

$$= \frac{2}{l}\int_0^l \psi(x) \sin\left(\frac{jN\pi x}{l}\right) dx$$

$$= \frac{2V}{l}\int_{\frac{Kl}{N}-\delta}^{\frac{Kl}{N}+\delta} \sin\left(\frac{jN\pi x}{l}\right) dx$$

$$= \frac{2V}{l}\left[\frac{-l}{jN\pi}\cos\left(\frac{jN\pi x}{l}\right)\right]\Bigg|_{\frac{Kl}{N}-\delta}^{\frac{Kl}{N}+\delta}.$$

Sec 5.1

#11, p.109 (cont.) Therefore

$$\frac{jN\pi c\, c_{jN}}{\ell} = \frac{2V}{jN\pi}\left[\cos\left(\overbrace{\frac{jN\pi}{\ell}\left(\frac{K\ell}{N}-\delta\right)}^{B}\right) - \cos\left(\overbrace{\frac{jN\pi}{\ell}\left(\frac{K\ell}{N}+\delta\right)}^{A}\right)\right]$$

$$= \frac{4V}{jN\pi}\sin\left(\overset{\frac{A+B}{2}}{\underbrace{j\pi K}_{\text{Integer multiple of }\pi}}\right)\sin\left(\overset{\frac{A-B}{2}}{\frac{jN\pi\delta}{\ell}}\right)$$

$$= 0 .$$

Consequently, $c_{jN} = 0$ for all $j = 1, 2, 3, \ldots$

Sec. 5.1

#11, p. 109. On a string with fixed ends, show that if the center of a hammer blow is exactly at a node of the n^{th} harmonic (i.e. at a place where the n^{th} eigenfunction vanishes), the n^{th} overtone is absent from the solution.

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$$(****') \qquad u_t(x,0) = \psi(x) = \begin{cases} V & \text{if } \left|x - \frac{K\ell}{N}\right| \leq \delta, \\ 0 & \text{otherwise.} \end{cases}$$

As in #10, p. 108, the series solution is

#11, p.109 (cont.)

$$(\dagger) \qquad u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi c t}{l}\right)$$

where $\dfrac{n\pi c c_n}{l} = n^{th}$ Fourier sine coefficient of ψ $(n = 1, 2, 3, \dots)$.

To show that the N^{th} overtone is absent from the solution, we must show that $c_N = 0$. In fact, we will show more, for we shall show that $0 = c_N = c_{2N} = c_{3N} = \dots$

Let $j \geq 1$ be an integer. Using the formula above for c_n with $n = jN$, and proceeding as in #10, p.108 we find

$$\frac{jN\pi c c_{jN}}{l} = (jN)^{th} \text{ Fourier sine coefficient of } \psi \text{ (given in } (\cancel{****})\text{)}$$

$$= \frac{2}{l} \int_0^l \psi(x) \sin\left(\frac{jN\pi x}{l}\right) dx$$

$$= \frac{2V}{l} \int_{\frac{Kl}{N} - \delta}^{\frac{Kl}{N} + \delta} \sin\left(\frac{jN\pi x}{l}\right) dx$$

$$= \frac{2V}{l} \left[\frac{-l}{jN\pi} \cos\left(\frac{jN\pi x}{l}\right) \right] \Bigg|_{\frac{Kl}{N} - \delta}^{\frac{Kl}{N} + \delta} .$$

Sec 5.1

#11, p.109 (cont.) Therefore

$$\frac{jN\pi c c_{jN}}{\ell} = \frac{2V}{jN\pi}\Big[\cos\Big(\overbrace{\frac{jN\pi}{\ell}\Big(\frac{K\ell}{N}-\delta\Big)}^{B}\Big) - \cos\Big(\overbrace{\frac{jN\pi}{\ell}\Big(\frac{K\ell}{N}+\delta\Big)}^{A}\Big)\Big]$$

$$= \frac{4V}{jN\pi}\sin\Big(\overset{\frac{A+B}{2}}{\underbrace{j\pi K}_{\substack{\text{Integer}\\ \text{multiple of }\pi}}}\Big)\sin\Big(\overset{\frac{A-B}{2}}{\frac{jN\pi\delta}{\ell}}\Big)$$

$$= 0 .$$

Consequently, $c_{jN} = 0$ for all $j = 1, 2, 3, \ldots$