Sec. 5.2, p. 113

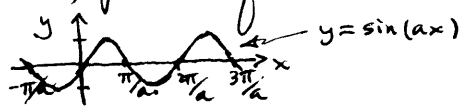
#1. For each of the following functions, state whether it is even, odd, or periodic. If periodic, what is its (fundamental) period?

(a) $\sin(ax)$ $(a>0)$ (d) $\tan(x^2)$

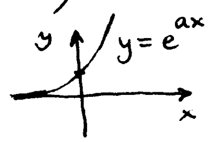
(b) e^{ax} $(a>0)$ (e) $|\sin(x/b)|$ $(b>0)$

(c) x^m (m an integer) (f) $x\cos(ax)$ $(a>0)$

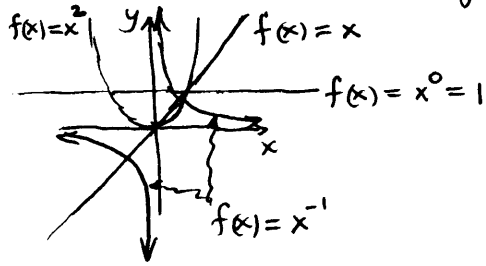
(a) $f(x)=\sin(ax)$ $(a>0)$ is an odd, periodic function with fundamental period $\frac{2\pi}{a}$.



(b) $f(x)=e^{ax}$ $(a>0)$ is neither even, nor odd, nor periodic.



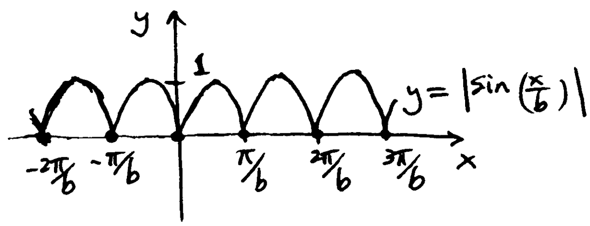
(c) $f(x)=x^m$ (m an integer) is either even or odd depending upon whether the integer m is even or odd; $f(x)=x^m$ is not periodic unless $m=0$.



(d) $f(x)=\tan(x^2)$ is an even, nonperiodic function:

$$f(-x)=\tan((-x)^2)=\tan(x^2)=f(x).$$

(e) $f(x)=|\sin(\frac{x}{b})|$ $(b>0)$ is an even periodic function with fundamental period $\frac{\pi}{b}$.



#1 (cont.) (f) $f(x) = x\cos(ax)$ $(a > 0)$ is an odd, nonperiodic function:

$$f(-x) = -x\cos(-ax) = -x\cos(ax) = -f(x).$$

#4. (a) Use

(5) $\displaystyle\int_{-l}^{l} (\text{odd})\,dx = 0$ and $\displaystyle\int_{-l}^{l} (\text{even})\,dx = 2\int_{0}^{l} (\text{even})\,dx$

to prove that if $\varphi = \varphi(x)$ is an odd function, its full Fourier series on $(-l, l)$ has only sine terms.

(b) Also, if $\varphi = \varphi(x)$ is an even function, its full (Fourier) series on $(-l, l)$ has only cosine terms.

(a) If φ is an odd function, then $f(x) = \varphi(x)\cos\left(\frac{n\pi x}{l}\right)$ is an odd function (since the product of an odd function and an even function is an odd function), so by the first part of (5),

$$a_n = \frac{1}{l}\int_{-l}^{l} \varphi(x)\cos\left(\frac{n\pi x}{l}\right)dx = 0 \quad \text{for all } n.$$

Thus the full Fourier series of φ has only sine terms:

$$\overset{0}{\cancel{a_0}} + \sum_{n=1}^{\infty}\left[\overset{0}{\cancel{a_n}}\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right] = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right).$$

(b) If φ is an even function then $f(x) = \varphi(x)\sin\left(\frac{n\pi x}{l}\right)$ is odd, so by (5)

$$b_n = \frac{1}{l}\int_{-l}^{l} \varphi(x)\sin\left(\frac{n\pi x}{l}\right)dx = 0 \quad \text{for all } n.$$

Therefore the full Fourier series of φ has only cosine terms.

#5. Show that the Fourier sine series of the function $\varphi = \varphi(x)$ on $(0, l)$ can be derived from the full Fourier series of $\tilde{\varphi} = \tilde{\varphi}(x)$, the odd extension φ to $(-l, l)$.

Given $\varphi = \varphi(x)$ on $(0, l)$, define

$$(*) \qquad \tilde{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \in (0,l), \\ 0 & \text{if } x = 0, \\ -\varphi(-x) & \text{if } x \in (-l, 0). \end{cases}$$

Clearly $(-l, l)$ is the domain of $\tilde{\varphi}$ and $\tilde{\varphi}$ is an extension of φ (i.e. $\tilde{\varphi}(x) = \varphi(x)$ for all $x \in (0, l)$, the domain of φ). Also it is not hard to see that $\tilde{\varphi}$ is an odd function:

$$\tilde{\varphi}(-x) = \begin{cases} \varphi(-x) & \text{if } -x \in (0,l), \\ 0 & \text{if } -x = 0, \\ -\varphi(-(-x)) & \text{if } -x \in (-l,0), \end{cases}$$

$$= \begin{cases} \varphi(-x) & \text{if } x \in (-l,0), \\ 0 & \text{if } x = 0 \\ -\varphi(x) & \text{if } x \in (0,l) \end{cases}$$

$$= -\tilde{\varphi}(x)$$

for all $x \in (-l, l)$. (The function $\tilde{\varphi}$ given by $(*)$ is called the odd extension of φ to the interval $(-l, l)$.)

By problem #4(a), the full Fourier series of $\tilde{\varphi}$ on $(-l, l)$ has only sine terms:

$$(\dagger) \qquad \text{The full Fourier series of } \tilde{\varphi} \text{ on } (-l,l) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

#5 (cont.) where, for $n = 1, 2, 3, \dots$,

$$b_n = \frac{\langle \tilde{\varphi}, \sin(\frac{n\pi \cdot}{\ell}) \rangle}{\langle \sin(\frac{n\pi \cdot}{\ell}), \sin(\frac{n\pi \cdot}{\ell}) \rangle}$$

$$= \frac{\int_{-\ell}^{\ell} \tilde{\varphi}(x) \sin\left(\frac{n\pi x}{\ell}\right) dx}{\int_{-\ell}^{\ell} \sin^2\left(\frac{n\pi x}{\ell}\right) dx}$$

$$= \frac{1}{\ell} \int_{-\ell}^{\ell} \tilde{\varphi}(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$$

$$= \frac{2}{\ell} \int_{0}^{\ell} \tilde{\varphi}(x) \sin\left(\frac{n\pi x}{\ell}\right) dx \qquad \text{(by (5) (see problem #4))}$$

$$= \frac{2}{\ell} \int_{0}^{\ell} \varphi(x) \sin\left(\frac{n\pi x}{\ell}\right) dx \qquad \left(\begin{array}{l}\text{because } \tilde{\varphi}(x) = \varphi(x) \\ \text{for all } x \in (0, \ell)\end{array}\right)$$

$$= \frac{\int_{0}^{\ell} \varphi(x) \sin\left(\frac{n\pi x}{\ell}\right) dx}{\int_{0}^{\ell} \sin^2\left(\frac{n\pi x}{\ell}\right) dx}$$

$$(\dagger\dagger) \qquad b_n = \frac{\langle \varphi, \sin(\frac{n\pi \cdot}{\ell}) \rangle}{\langle \sin(\frac{n\pi \cdot}{\ell}), \sin(\frac{n\pi \cdot}{\ell}) \rangle}$$

But $(\dagger\dagger)$ is precisely the formula for the nth Fourier sine coefficient of φ on $(0, \ell)$. Therefore $(\dagger)$ is also the Fourier sine series of φ on $(0, \ell)$.

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#6 Show that the cosine series on $(0, \ell)$ can be derived from the full series on $(-\ell, \ell)$ by using the even extension of a function.

Let φ be any integrable function on $(0,\ell)$ and let $\tilde{\varphi}$ be its even extension : $\tilde{\varphi}(x) = \begin{cases} \varphi(-x) & \text{if } -\ell < x < 0, \\ \varphi(x) & \text{if } 0 < x < \ell. \end{cases}$ $\quad (*)$

By problem # 4(b), $\tilde{B}_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \tilde{\varphi}(x)\sin\left(\frac{n\pi x}{\ell}\right)dx = 0$ for $n = 1, 2, 3, \ldots$.

Also, $\tilde{A}_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \underbrace{\tilde{\varphi}(x)}_{\text{even}} \underbrace{\cos\left(\frac{n\pi x}{\ell}\right)}_{\text{even}} dx = \frac{2}{\ell}\int_0^{\ell} \tilde{\varphi}(x)\cos\left(\frac{n\pi x}{\ell}\right)dx$ (see (5) p. 111)

$$= \frac{2}{\ell}\int_0^{\ell} \varphi(x)\cos\left(\frac{n\pi x}{\ell}\right)dx \quad (\text{by } (*))$$

$$= A_n$$

for $n = 0, 1, 2, \ldots$. Therefore the cosine series of φ on $(0,\ell)$ is

$$\varphi(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right) = \underbrace{\frac{\tilde{A}_0}{2} + \sum_{n=1}^{\infty}\left[\tilde{A}_n\cos\left(\frac{n\pi x}{\ell}\right) + \tilde{B}_n \sin\left(\frac{n\pi x}{\ell}\right)\right]}_{\text{full Fourier series of the even extension } \tilde{\varphi} \text{ on } (-\ell,\ell).}$$

#14 Find the full Fourier series of $|x|$ on $(-\ell, \ell)$ in its real and complex forms.

#8. (a) Prove that differentiation switches even functions to odd ones, and odd functions to even ones.

(b) Prove the same for integration provided that we ignore the constant of integration.

(a) Let f be a differentiable even (i.e. $f(-x) \overset{(*)}{=} f(x)$ for all $x \in \mathrm{dom}(f)$) function. For each $x_0 \in \mathrm{dom}(f')$

$$f'(-x_0) = \lim_{h \to 0} \frac{f(-x_0+h) - f(-x_0)}{h} = \lim_{h \to 0} \frac{f(x_0-h) - f(x_0)}{h} \qquad (\text{by } (*))$$

$$= -\lim_{h \to 0} \frac{f(x_0-h) - f(x_0)}{-h}$$

$$= -f'(x_0).$$

That is, f' is odd ($f'(-x) = -f'(x)$ for all $x \in \mathrm{dom}(f')$).

[The proof when f is a differentiable odd function is similar.]

(b) Let f be an even integrable function on $[-a,a]$, and define the (indefinite) integral of f by

$$(**) \qquad F(x) = \int_0^x f(t)\,dt, \qquad x \in [-a,a].$$

If $x \in [-a,a]$ then

$$F(-x) = \int_0^{-x} f(t)\,dt \qquad (\text{by } (**))$$

$$= -\int_0^x f(-u)\,du \qquad (\text{by the change of variables } t \mapsto -u)$$

$$= -\int_0^x f(u)\,du \qquad (\text{by evenness of } f)$$

$$= -F(x) \qquad (\text{by } (**)).$$

That is the integral of f is an odd function. [The case f odd is similar.]

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#10. (a) Let φ be a continuous function on $(0,\ell)$. Under what conditions is its odd extension also a continuous function?

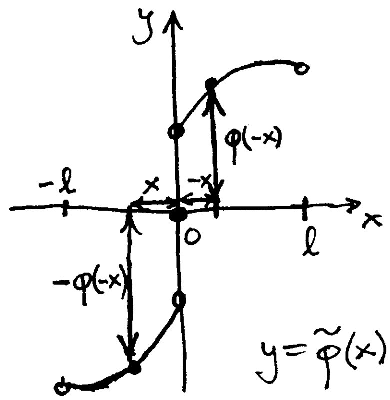
(b) Let φ be a differentiable function on $(0,\ell)$. Under what conditions is its odd extension also a differentiable function?

(c) Same as part (a) for the even extension.

(d) " " " (b) " " " " .

(a) The odd extension $\tilde{\varphi}$ of φ is given by

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \in (0,\ell), \\ -\varphi(-x) & \text{if } x \in (-\ell,0). \end{cases}$$



[See the diagram for the case $x \in (-\ell,0)$.]

Since $\tilde{\varphi}$ is odd on $(-\ell,\ell)$ [i.e. $\tilde{\varphi}(-x) = -\tilde{\varphi}(x)$ for all $x \in (-\ell,\ell)$] it follows that $-\tilde{\varphi}(0) = \tilde{\varphi}(-0) = \tilde{\varphi}(0)$. Hence $\tilde{\varphi}(0) = 0$ because that is the only solution to the equation $-y = y$. But if $\tilde{\varphi}$ is continuous at 0 then $\lim_{x\to 0^+} \varphi(x) = \lim_{x \to 0^+} \tilde{\varphi}(x) = \tilde{\varphi}(0) = 0$. Thus

$$\boxed{\lim_{x\to 0^+} \varphi(x) = 0}$$

is necessary (and also sufficient) for $\tilde{\varphi}$ to be continuous on $(-\ell,\ell)$.

(b) Let $\tilde{\varphi}'_+$ and $\tilde{\varphi}'_-$ denote the right hand and left hand derivative

#10 (b) (cont.) functions of $\tilde{\varphi}$. In order for $\tilde{\varphi}$ to be differentiable on $(-\ell,\ell)$ it is necessary (and also sufficient) that $\tilde{\varphi}_+'(0)$ and $\tilde{\varphi}_-'(0)$ exist and are equal. But

$$\tilde{\varphi}_+'(0) = \lim_{h\to 0^+} \frac{\tilde{\varphi}(h) - \tilde{\varphi}(0)}{h} = \lim_{h\to 0^+} \frac{\tilde{\varphi}(h)}{h} = \lim_{h\to 0^+} \frac{\varphi(h)}{h}$$

and

$$\tilde{\varphi}_-'(0) = \lim_{h\to 0^-} \frac{\tilde{\varphi}(h) - \tilde{\varphi}(0)}{h} = \lim_{h\to 0^-} \frac{\tilde{\varphi}(h)}{h} = \lim_{h\to 0^-} \frac{-\varphi(-h)}{h}$$

$$= \lim_{h\to 0^+} \frac{\varphi(h)}{h}.$$

Therefore, the existence of the limit

$$\boxed{\lim_{h\to 0^+} \frac{\varphi(h)}{h}}$$

is necessary (and also sufficient) for differentiability of $\tilde{\varphi}$ on $(-\ell,\ell)$.

(c) By similar techniques, the existence of the limit

$$\varphi(0^+) = \boxed{\lim_{x\to 0^+} \varphi(x)}$$

is necessary (and also sufficient) for continuity of the even extension of φ on $(-\ell,\ell)$.

(d) By similar techniques,

$$\boxed{\lim_{h\to 0^+} \frac{\varphi(h) - \varphi(0^+)}{h} = 0}$$

is necessary (and also sufficient) for differentiability of the even extension of φ on $(-\ell,\ell)$.

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#11. Find the full Fourier series of $f(x)=e^x$ on $(-l,l)$ in its real and complex form.

We compute the complex form first:

$$\hat{f}(n) = \frac{\langle f, e^{\frac{in\pi\cdot}{l}} \rangle}{\langle e^{\frac{in\pi\cdot}{l}}, e^{\frac{in\pi\cdot}{l}} \rangle} = \frac{\int_{-l}^{l} f(x) e^{-\frac{in\pi x}{l}}\,dx}{\int_{-l}^{l} e^{\frac{in\pi x}{l}} \cdot e^{-\frac{in\pi x}{l}}\,dx} = \frac{1}{2l}\int_{-l}^{l} e^{x} e^{-\frac{in\pi x}{l}}dx$$

$$= \frac{1}{2l}\int_{-l}^{l} e^{\left(1-\frac{in\pi}{l}\right)x}dx = \frac{1}{2l}\left(\frac{e^{\left(1-\frac{in\pi}{l}\right)x}}{1-\frac{in\pi}{l}}\right)\Bigg|_{-l}^{l} = \frac{1}{2(l-in\pi)}\left[e^{l}\cdot e^{-in\pi} - e^{-l}\cdot e^{in\pi}\right].$$

Using $e^{\pm in\pi} = \cos(n\pi) \pm i\sin(n\pi) = (-1)^n$, it follows that $\hat{f}(n) = \frac{(-1)^n}{l-in\pi}\cdot\frac{(e^{l}-e^{-l})}{2}$ $= \frac{(-1)^n \sinh(l)}{l-in\pi}$ for $n=0,\pm1,\pm2,\ldots$ The full Fourier series of $f(x)=e^x$ on $(-l,l)$ in complex form is

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{\frac{in\pi x}{l}} = \boxed{\sinh(l)\sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{\frac{in\pi x}{l}}}{l-in\pi}}.$$

We now get the real form of the full Fourier series of f on $(-l,l)$ from the complex form via the Euler identity: $e^{i\theta} = \cos\theta + i\sin\theta$ for real θ.

$$\sinh(l)\sum_{n=-\infty}^{\infty}\frac{(-1)^n e^{\frac{in\pi x}{l}}}{l-in\pi} = \sinh(l)\left\{\underbrace{\frac{1}{l}}_{n=0} + \sum_{m=1}^{\infty}\left[\underbrace{\frac{(-1)^m e^{\frac{im\pi x}{l}}}{l-im\pi}}_{n=m\geq1} + \underbrace{\frac{(-1)^{-m} e^{-\frac{im\pi x}{l}}}{l+im\pi}}_{n=-m\leq-1}\right]\right\}$$

$$= \sinh(l)\left\{\frac{1}{l} + \sum_{m=1}^{\infty}\frac{(l+im\pi)(-1)^m\left[\cos\left(\frac{m\pi x}{l}\right)+i\sin\left(\frac{m\pi x}{l}\right)\right] + (l-im\pi)(-1)^m\left[\cos\left(\frac{m\pi x}{l}\right) - i\sin(\;\right.}{(l-im\pi)(l+im\pi)}\right.$$

#11 (cont.) Therefore the real form of the full Fourier series of f on $(-\ell,\ell)$ is

$$\sinh(\ell)\left\{ \frac{1}{\ell} + \sum_{m=1}^{\infty} \frac{(-1)^m(\ell+\cancel{im\pi}+\ell-\cancel{im\pi})\cos(m\pi x/\ell) + i(-1)^m(\cancel{\ell}+im\pi-\cancel{\ell}+im\pi)\sin(m\pi x/\ell)}{\ell^2+m^2\pi^2} \right\}$$

$$= \boxed{\sinh(\ell)\left\{ \frac{1}{\ell} + \sum_{m=1}^{\infty}\left[\frac{(-1)^m 2\ell}{\ell^2+m^2\pi^2}\cos\left(\frac{m\pi x}{\ell}\right) + \frac{(-1)^{m+1}2m\pi}{\ell^2+m^2\pi^2}\sin\left(\frac{m\pi x}{\ell}\right)\right]\right\}}$$

#12. Repeat exercise #11 for $f(x) = \cosh(x)$ on $(-\ell,\ell)$.

Observe first that, given an integrable function h on $(-\ell,\ell)$, if we let

$$g(x) = h(-x)$$

for all $x \in (-\ell,\ell)$, then $\hat{g}(n) = \hat{h}(-n)$ for all $n = 0, \pm 1, \pm 2, \ldots$ [To see this, we compute:

$$\hat{g}(n) = \frac{1}{2\ell}\int_{-\ell}^{\ell} g(x)e^{\frac{-in\pi x}{\ell}}\,dx = \frac{1}{2\ell}\int_{-\ell}^{\ell} h(-x)e^{\frac{-in\pi x}{\ell}}\,dx = \frac{1}{2\ell}\int_{-\ell}^{\ell} h(x)e^{\frac{in\pi x}{\ell}}\,dx = \hat{h}(-n).]$$

Applying this with $h(x) = e^x$ on $(-\ell,\ell)$, problem #11 yields

$$e^x \sim \sum_{n=-\infty}^{\infty} \hat{h}(n)e^{\frac{in\pi x}{\ell}} = \sum_{n=-\infty}^{\infty} \frac{\sinh(\ell)(-1)^n}{\ell - in\pi}e^{\frac{in\pi x}{\ell}},$$

$$e^{-x} \sim \sum_{n=-\infty}^{\infty} \hat{h}(-n)e^{\frac{in\pi x}{\ell}} = \sum_{n=-\infty}^{\infty} \frac{\sinh(\ell)(-1)^{-n}}{\ell + in\pi}e^{\frac{in\pi x}{\ell}}.$$

#12 (cont.) Therefore, the complex form of the full Fourier series of $f(x) = \cosh(x) = \dfrac{e^{x}+e^{-x}}{2}$ on $(-l, l)$ is:

$$\frac{e^{x}+e^{-x}}{2} \sim \frac{1}{2}\sum_{n=-\infty}^{\infty} \sinh(l)(-1)^{n}\left[\frac{1}{l-in\pi}+\frac{1}{l+in\pi}\right]e^{\frac{in\pi x}{l}}$$

$$\frac{e^{x}+e^{-x}}{2} \sim \frac{1}{2}\sum_{n=-\infty}^{\infty} \sinh(l)(-1)^{n}\left[\frac{l+\cancel{in\pi}+l-\cancel{in\pi}}{(l-in\pi)(l+in\pi)}\right]e^{\frac{in\pi x}{l}}$$

$$\frac{e^{x}+e^{-x}}{2} \sim \boxed{\sum_{n=-\infty}^{\infty} \frac{l\sinh(l)(-1)^{n}}{l^{2}+n^{2}\pi^{2}}\,e^{\frac{in\pi x}{l}}}\,.$$

The real form is obtained from the complex form via the Euler identity as in #11:

$$\frac{e^{x}+e^{-x}}{2} \sim \overbrace{\frac{\sinh(l)}{l}}^{n=0} + \sum_{m=1}^{\infty}\left\{\overbrace{\frac{l\sinh(l)(-1)^{m}\left[\cos\left(\frac{m\pi x}{l}\right)+i\sin\left(\frac{m\pi x}{l}\right)\right]}{l^{2}+m^{2}\pi^{2}}}^{n=m\geq 1} + \overbrace{\frac{l\sinh(l)(-1)^{-m}\left[\cos\left(\frac{m\pi x}{l}\right)-i\sin(\right.}{l^{2}+(-m)^{2}\pi^{2}}}^{n=-m\leq -1}\right.$$

$$\frac{e^{x}+e^{-x}}{2} \sim \frac{\sinh(l)}{l} + l\sinh(l)\sum_{m=1}^{\infty}\frac{(-1)^{m}\left[\cos\left(\frac{m\pi x}{l}\right)+\cancel{i\sin\left(\frac{m\pi x}{l}\right)}+\cos\left(\frac{m\pi x}{l}\right)-\cancel{i\sin\left(\frac{m\pi x}{l}\right)}\right]}{l^{2}+m^{2}\pi^{2}}$$

$$\frac{e^{x}+e^{-x}}{2} \sim \boxed{\frac{\sinh(l)}{l} + l\sinh(l)\sum_{m=1}^{\infty}\frac{2(-1)^{m}}{l^{2}+m^{2}\pi^{2}}\cos\left(\frac{m\pi x}{l}\right)}$$

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#15. Without any computation, predict which of the Fourier coefficients of $f(x) = |\sin x|$ on the interval $(-\pi, \pi)$ must vanish.

Note first that f is an even function:

$$f(-x) = |\sin(-x)| = |-\sin(x)| = |\sin(x)| = f(x) \quad \text{for all } x \in [-\pi, \pi].$$

Therefore, by exercise #4(b), the Fourier sine coefficients of f vanish:

$$\boxed{b_n = \frac{\langle f, \sin(n\cdot)\rangle}{\langle \sin(n\cdot), \sin(n\cdot)\rangle} = 0}$$

for $n = 1, 2, 3, \ldots$ Furthermore, since f is even, its cosine coefficients are given by

$$a_n = \frac{\langle f, \cos(n\cdot)\rangle}{\langle \cos(n\cdot), \cos(n\cdot)\rangle} = \frac{\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx}{\int_{-\pi}^{\pi} \cos^2(nx)\,dx} \overset{(\text{if } n \neq 0)}{=} \frac{2}{\pi}\int_0^{\pi} |\sin(x)|\cos(nx)\,dx,$$

and since $\sin(x) > 0$ on $(0, \pi)$, it follows that

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx, \qquad (n = 1, 2, 3, \ldots).$$

We claim that the odd-index cosine coefficients vanish:

$$\boxed{a_{2k+1} = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos((2k+1)x)\,dx,} \qquad (k = 0, 1, 2, \ldots).$$

To show this (without computation) we will make use of the following fact:

If g is integrable on $[k-a, k+a]$ and $g(k+x) = -g(k-x)$ for all $x \in [-a, a]$, then $\int_{k-a}^{k+a} g(x)\,dx = 0$.

#15 (cont.) Note that if $k=0$, this fact reduces to a well-known statement: if g is an odd function then $\int_{-a}^{a} g(x)\,dx = 0$. The proof of the fact follows along the same lines of reasoning as the result about odd functions having integral zero on $[-a,a]$. The details of the proof are left to the reader.

Consider the function $g(x) = \sin(x)\cos((2k+1)x)$ where k is a fixed nonnegative integer. Using the identities

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$

and the values $\sin((2k+1)\frac{\pi}{2}) = (-1)^k$ and $\cos((2k+1)\frac{\pi}{2}) = 0$, one easily shows that $g(\frac{\pi}{2}+x) = -g(\frac{\pi}{2}-x)$ for all real x. Applying the fact at the bottom of the previous page (with $k = \pi/2 = a$) yields

$$\int_0^{\pi} g(x)\,dx = 0 .$$

That is,

$$a_{2k+1} = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos((2k+1)x)\,dx = \frac{2}{\pi}\int_0^{\pi} g(x)\,dx = 0 .$$

#14 (cont.) By problem #4(b), the even function $|x|$ satisfies

$$B_n = \frac{1}{\ell}\int_{-\ell}^{\ell} |x| \sin\left(\frac{n\pi x}{\ell}\right) dx = 0 \qquad \text{for } n = 1, 2, 3, \ldots$$

Also, using (5) on p. 111, for $n = 1, 2, 3, \ldots$ we have

$$A_n = \frac{1}{\ell}\int_{-\ell}^{\ell} |x| \cos\left(\frac{n\pi x}{\ell}\right) dx = \frac{2}{\ell}\int_0^{\ell} |x| \cos\left(\frac{n\pi x}{\ell}\right) dx$$

$$= \frac{2}{\ell}\int_0^{\ell} \overbrace{x}^{U} \overbrace{\cos\left(\frac{n\pi x}{\ell}\right) dx}^{dV} \overset{\text{parts.}}{=} \frac{2}{\ell}(x)\left[\frac{\ell}{n\pi}\sin\left(\frac{n\pi x}{\ell}\right)\right]\Bigg|_0^{\ell} - \frac{2}{\ell}\left(\frac{\ell}{n\pi}\right)\int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) dx$$

$$= -\frac{2}{n\pi}\int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) dx = +\frac{2}{n\pi}\left[\frac{\ell}{n\pi}\cos\left(\frac{n\pi x}{\ell}\right)\right]\Bigg|_0^{\ell} = \frac{2\ell}{(n\pi)^2}\big(\underbrace{(-1)^n - 1}_{\substack{-2 \text{ if } n \text{ odd} \\ 0 \text{ if } n \text{ even}}}\big).$$

$$A_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} |x| \cdot 1\, dx = \frac{2}{\ell}\int_0^{\ell} |x|\, dx = \frac{2}{\ell}\int_0^{\ell} x\, dx = \frac{2}{\ell}\left(\frac{\ell^2}{2}\right) = \ell.$$

$$|x| \sim \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi x}{\ell}\right) + B_n \sin\left(\frac{n\pi x}{\ell}\right)\right]$$

$$|x| \sim \frac{\ell}{2} + \sum_{n=1}^{\infty} \frac{2\ell\,\overbrace{[(-1)^n - 1]}^{\substack{0 \text{ if } n \text{ even} \\ -2 \text{ if } n \text{ odd}}}}{n^2\pi^2}\cos\left(\frac{n\pi x}{\ell}\right)$$

$$\boxed{|x| \sim \frac{\ell}{2} - \frac{4\ell}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}\cos\left(\frac{(2k-1)\pi x}{\ell}\right)}$$

$\boxed{\text{Real Form}}$

#14 (cont.) If we substitute in the real form for the full Fourier series for $|x|$ using $\cos(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ with $\theta = \dfrac{(2k-1)\pi x}{l}$, we obtain

$$|x| \sim \frac{l}{2} - \frac{4l}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{2(2k-1)^2} \left[e^{i(2k-1)\pi x/l} + e^{-i(2k-1)\pi x/l} \right]$$

$$\boxed{|x| \sim \frac{l}{2} - \frac{2l}{\pi^2} \sum_{m=-\infty}^{\infty} \frac{1}{(2m-1)^2} e^{i(2m-1)\pi x/l}}$$

Complex Form