Sec. 5.5, pp. 139–140.

#2. Prove the Schwarz inequality (for any pair of real square-integrable functions):

$$(*) \qquad |(f,g)| \le \|f\| \cdot \|g\|.$$

If $g = 0$ then clearly equality holds in $(*)$:

$$|(f,g)| = 0 = \|f\| \cdot \|g\|.$$

Therefore we may suppose that $g \neq 0$, and consequently that

$$\|g\|^2 = (g,g) = \int_a^b [g(t)]^2 dt > 0.$$

Consider the nonnegative quadratic function of t given by

$$\begin{aligned} H(t) &= \|f - tg\|^2 \\ &= (f - tg, f - tg) \\ &= (f,f) - 2t(f,g) + t^2(g,g). \end{aligned}$$

Then $0 = H'(t) = -2(f,g) + 2t(g,g)$ implies $t = \dfrac{(f,g)}{(g,g)}$. Since $H''(t) = 2(g,g) > 0$ for all t, the critical number $t = (f,g)/(g,g)$ minimizes H:

$$0 \le H\left(\frac{(f,g)}{(g,g)}\right) = (f,f) - 2\frac{(f,g)}{(g,g)} \cdot (f,g) + \left(\frac{(f,g)}{(g,g)}\right)^2 \cdot (g,g) = (f,f) - \frac{(f,g)^2}{(g,g)}$$

so rearranging we have $(f,g)^2 \le (f,f) \cdot (g,g)$. Extracting square roots of both sides of this inequality gives $(*)$:

$$|(f,g)| = \sqrt{(f,g)^2} \le \sqrt{(f,f)} \cdot \sqrt{(g,g)} = \|f\| \cdot \|g\|.$$

Sec. 5.5, pp. 139-140.

#4. (a) Solve the problem

$$u_t - ku_{xx} = 0 \quad \text{for} \quad 0<x<\ell,\ 0<t<\infty,$$

$$u(x,0) = \varphi(x) \quad \text{for} \quad 0\le x\le \ell,$$

with the unusual boundary conditions

$$u_x(0,t) = u_x(\ell,t) = \frac{u(\ell,t)-u(0,t)}{\ell} \quad \text{for} \quad 0\le t<\infty.$$

Assume that there are no negative eigenvalues.

(b) Assuming that you can take limits term by term (in the infinite series resulting from part (a)) show that

$$\lim_{t\to\infty} u(x,t) = A + Bx .$$

(c) Use Green's first identity and exercise #3 to show that there are no negative eigenvalues.

(d) Find A and B.

(a) If we separate variables via $u(x,t) = X(x)T(t)$ in this problem, we arrive at the coupled system

$$\begin{cases} T' + \lambda k T = 0 , \\ X'' + \lambda X = 0 , \qquad X'(0) = X'(\ell) = \dfrac{X(\ell)-X(0)}{\ell} . \end{cases}$$

It is easy to verify that $\lambda = 0$ is an eigenvalue and any (nonzero) linear function $X(x) = Ax+B$ is a corresponding eigenfunction.

Suppose $\lambda = \beta^2 > 0$. The general solution to the ODE is $X(x) = c_1\cos(\beta x) + c_2\sin(\beta x)$.

Applying the B.C.'s gives

$$0 = X'(0) - X'(\ell) = \beta c_2 - \beta c_2\cos(\beta\ell) + \beta c_1 \sin\beta\ell$$

$$0 = X'(0) - \frac{X(\ell)-X(0)}{\ell} = \beta c_2 - \frac{c_1\cos(\beta\ell)+c_2\sin(\beta\ell) - c_1}{\ell}$$

The existence of a nontrivial solution to this linear homogeneous system of

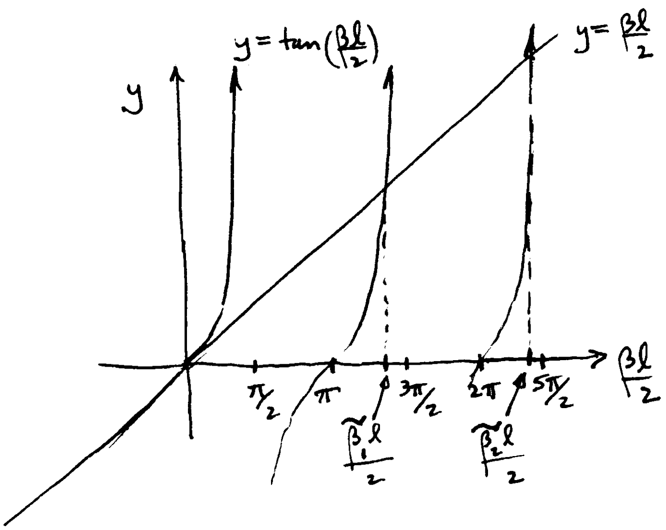
#4 (a) (cont.) equations in the variables c_1 and c_2 is equivalent to the vanishing of the determinant of the coefficient matrix:

$$\begin{cases} 0 = (1-\cos\beta l)\,c_2 + \sin(\beta l)\,c_1 \\ 0 = (\beta l - \sin\beta l)\,c_2 + (1-\cos\beta l)\,c_1 \end{cases}$$

$$0 = \begin{vmatrix} \sin(\beta l) & 1-\cos(\beta l) \\ 1-\cos(\beta l) & \beta l - \sin\beta l \end{vmatrix} = \beta l \sin(\beta l) - \sin^2(\beta l) - 1 + 2\cos(\beta l) - \cos^2\beta l$$

$$0 = \beta l \sin(\beta l) - 2 + 2\cos\beta l \iff \beta l \sin(\beta l) = 2(1-\cos\beta l).$$

$$\Rightarrow \begin{cases} \text{either } \beta l = 2n\pi, \quad (n=1,2,3,\dots) \\ \text{or} \\ \dfrac{\beta l}{2} = \dfrac{1-\cos(\beta l)}{\sin(\beta l)} = \tan\left(\dfrac{\beta l}{2}\right). \end{cases}$$



In the first case we have

$$\beta_n = \frac{2n\pi}{l} \quad (n=1,2,3,\dots)$$

and in the second case

$$\tilde\beta_n \in \left(\frac{2n\pi}{l}, (2n+1)\frac{\pi}{l}\right) \quad (n=1,2,3,\dots)$$

with $\lim_{n\to\infty}\left[\tilde\beta_n - (2n+1)\frac{\pi}{l}\right] = 0.$

In the first case, the corresponding eigenfunctions are (up to a constant multiple) $X_n(x) = \cos(\beta_n x) = \cos\left(\frac{2n\pi x}{l}\right)$, $(n=1,2,3,\dots)$, and in the second they are (up to a constant multiple) $\tilde X_n(x) = \tilde\beta_n l\cos(\tilde\beta_n x) - 2\sin(\tilde\beta_n x)$, $(n=1,2,3,\dots)$. Thus a (formal) solution is

$$u(x,t) = Ax + B + \sum_{n=1}^{\infty} c_n e^{-(\beta_n^2 k)t}\cos(\beta_n x) + \sum_{n=1}^{\infty} \tilde c_n e^{-(\tilde\beta_n^2 k)t}\left[\tilde\beta_n l\cos(\tilde\beta_n x) - 2\sin(\tilde\beta_n x)\right]$$

#4 (a) (cont.) where the coefficients c_n and $\tilde{c}_n$ $(n=1,2,3,\dots)$ are chosen so that

$$\varphi(x) = u(x,0) = Ax+B + \sum_{n=1}^{\infty} c_n \cos(\beta_n x) + \sum_{n=1}^{\infty} \tilde{c}_n \left[\tilde{\beta}_n \ell \cos(\tilde{\beta}_n x) - 2\sin(\tilde{\beta}_n x)\right]$$

for all $0 \le x \le \ell$.

(b) Taking the formal limit (i.e. term by term in the infinite series expression for $u = u(x,t)$), we have

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty}\left\{ Ax+B + \sum_{n=1}^{\infty} c_n \cancelto{0}{e^{-(\beta_n^2 k)t}} \cos(\beta_n x) + \sum_{n=1}^{\infty} \tilde{c}_n \cancelto{0}{e^{-(\tilde{\beta}_n^2 k)t}} \tilde{X}_n(x)\right\}$$

$$= Ax+B.$$

(c) Let $\lambda = -\beta^2 < 0$ and let $X = X(x)$ be a solution to

$$\begin{cases} X'' + \lambda X = X'' - \beta^2 X = 0 & \text{for } 0<x<\ell, \quad (*) \\ X'(0) = X'(\ell) = \dfrac{X(\ell)-X(0)}{\ell}. & (**) \end{cases}$$

By Green's first identity (exercise #12, Sec. 5.3) or integration by parts

$$(***) \quad \int_0^\ell X''(x) X(x)\,dx = -\int_0^\ell [X'(x)]^2 dx + X'(x)X(x)\Big|_0^\ell .$$

But $\displaystyle\int_0^\ell X''(x)X(x)\,dx = \beta^2 \int_0^\ell [X(x)]^2 dx$ by $(*)$,

$$-\int_0^\ell [X'(x)]^2 dx \le -\frac{[X(\ell)-X(0)]^2}{\ell} \quad \text{by exercise \#3,}$$

and $\displaystyle X'(x)X(x)\Big|_0^\ell = X'(\ell)X(\ell) - X'(0)X(0) = \left(\frac{X(\ell)-X(0)}{\ell}\right)(X(\ell)-X(0))$ by $(**)$.

#4 (c) (cont.) Substituting these expressions into (***) gives

$$\beta^2 \int_0^\ell [X(x)]^2 dx \le -\frac{[X(\ell)-X(0)]^2}{\ell} + \frac{[X(\ell)-X(0)]^2}{\ell} \le 0 .$$

Dividing by the positive quantity β^2 we have

$$\underbrace{0 \le}_{\text{Clear}} \int_0^\ell [X(x)]^2 dx \underbrace{\le 0}_{\text{From above}} \quad \text{and hence} \quad \int_0^\ell [X(x)]^2 dx = 0 .$$

By continuity of X on $(0,\ell)$ it then follows $X(x) = 0$ for all $0<x<\ell$, i.e. $\lambda = -\beta^2 < 0$ is <u>not</u> an eigenvalue of (*)-(**).

(d) By Theorem 1, Sec. 5.3, the eigenfunctions $Ax+B$, $\{\cos(\beta_n x)\}_{n=1}^\infty$, and $\{\tilde\beta_n \ell \cos(\tilde\beta_n x) - 2\sin(\tilde\beta_n x)\}_{n=1}^\infty$ of the eigenvalue problem (*)-(**) in part (c) are orthogonal on $(0,\ell)$. Therefore the constants in the expansion

$$(\dagger) \quad \varphi(x) = u(x,0) = Ax+B + \sum_{n=1}^\infty c_n \cos(\beta_n x) + \sum_{n=1}^\infty \tilde c_n \left[\tilde\beta_n \ell \cos(\tilde\beta_n x) - 2\sin(\tilde\beta_n x)\right]$$

of φ on $(0,\ell)$ are the Fourier coefficients of φ with respect to the orthogonal set of eigenfunctions.

The two-dimensional space of eigenfunctions corresponding to the eigenvalue $\lambda = 0$, $\{Ax+B :\ A, B \text{ are real numbers}\}$, has an orthogonal basis 1 and $x - \frac{\ell}{2}$ on $(0,\ell)$. Therefore, the particular choice of constants A and B in the expansion $(\dagger)$ of φ satisfies

$$(\dagger\dagger) \qquad Ax + B = A\left(x - \frac{\ell}{2}\right) + B' \cdot 1 \quad \text{for} \quad 0<x<\ell$$

Sec. 5.5, pp. 139–140.

#4 (d) (cont.) where A and B' are given by the familiar Fourier coefficient formulas:

$$A = \frac{(\varphi, x-\frac{\ell}{2})}{(x-\frac{\ell}{2}, x-\frac{\ell}{2})} \quad \text{and} \quad B' = \frac{(\varphi, 1)}{(1, 1)} .$$

But $(1,1) = \int_0^\ell 1^2 dx = \ell$

and $(x-\frac{\ell}{2}, x-\frac{\ell}{2}) = \int_0^\ell (x-\frac{\ell}{2})^2 dx = \frac{1}{3}(x-\frac{\ell}{2})^3 \Big|_0^\ell = \frac{1}{3}(\frac{\ell}{2})^3 - \frac{1}{3}(-\frac{\ell}{2})^3 = \frac{\ell^3}{12}$,

so

$$(\dagger\dagger\dagger) \quad \boxed{A = \frac{12}{\ell^3}\int_0^\ell \varphi(x)(x-\frac{\ell}{2})dx} \quad \text{and} \quad B' = \frac{1}{\ell}\int_0^\ell \varphi(x)dx .$$

From $(\dagger\dagger)$ it is clear that

$$B = B' - \frac{\ell}{2}A,$$

so substituting from $(\dagger\dagger\dagger)$ we have

$$B = \frac{1}{\ell}\int_0^\ell \varphi(x)dx - \frac{6}{\ell^2}\int_0^\ell \varphi(x)(x-\frac{\ell}{2})dx$$

$$B = \frac{2}{\ell^2}\int_0^\ell \varphi(x)\frac{\ell}{2}dx - \frac{2}{\ell^2}\int_0^\ell \varphi(x)(3x-\frac{3\ell}{2})dx$$

$$\boxed{B = \frac{2}{\ell^2}\int_0^\ell \varphi(x)(2\ell-3x)dx} .$$

Sec. 5.5, pp. 139–140.

#8 Prove that both integrals in

(12) $$\int_0^{\pi} g_+(\theta)\sin\left[(N+\tfrac{1}{2})\theta\right]d\theta + \int_{-\pi}^{0} g_-(\theta)\sin\left[(N+\tfrac{1}{2})\theta\right]d\theta$$
p. 135

tend to zero (as N tends to infinity).

Because the quotient of two piecewise continuous functions is piecewise continuous as long as the denominator isn't zero, it follows that

$$g_+(\theta) = \frac{f(x+\theta)-f(x^+)}{\sin(\theta/2)}$$

is piecewise continuous for $\theta \in (0,\pi]$. Also, l'Hospital's rule implies

$$\lim_{\theta\to 0^+} g_+(\theta) = \lim_{\theta\to 0^+} \frac{f(x+\theta)-f(x^+)}{\sin(\theta/2)} = \lim_{\theta\to 0^+} \frac{f'(x+\theta)}{\frac{1}{2}\cos(\theta/2)}$$

$$= 2f'(x^+).$$

Therefore, the discontinuity in g_+ at $\theta = 0$ is removable, and consequently $\int_0^{\pi} |g_+(\theta)|^2 d\theta < \infty$. If we apply Bessel's inequality (18), p. 128, with $X_n(\theta) = \sin\left[(n+\frac{1}{2})\theta\right]$, $(a,b) = (0,\pi)$, we get

$$(*) \qquad \sum_{n=1}^{\infty} A_n^2 \int_0^{\pi} |X_n(\theta)|^2 d\theta \le \int_0^{\pi} |g_+(\theta)|^2 d\theta$$

where $$A_n = \frac{(g_+, X_n)}{(X_n, X_n)} = \frac{\int_0^{\pi} g_+(\theta)\sin((n+\frac{1}{2})\theta)\,d\theta}{\int_0^{\pi} \sin^2((n+\frac{1}{2})\theta)\,d\theta} = \frac{2}{\pi}\int_0^{\pi} g_+(\theta)\sin((n+\tfrac{1}{2})\theta)d\theta$$

#8 (cont.) Substituting these expressions into Bessel's inequality (*) gives

$$\sum_{n=1}^{\infty} \left\{ \frac{2}{\pi} \int_0^{\pi} g_+(\theta) \sin((n+\tfrac{1}{2})\theta)\, d\theta \right\}^2 \cdot \frac{\pi}{2} \le \int_0^{\pi} |g_+(\theta)|^2 d\theta < \infty,$$

and consequently

$$\sum_{n=1}^{\infty} \left\{ \int_0^{\pi} g_+(\theta) \sin((n+\tfrac{1}{2})\theta)\, d\theta \right\}^2 < \infty .$$

Since the terms in a convergent series must go to zero as $n \to \infty$, we get the desired result:

$$\lim_{n\to\infty} \left\{ \int_0^{\pi} g_+(\theta) \sin((n+\tfrac{1}{2})\theta)\, d\theta \right\}^2 = 0$$

and hence

$$\lim_{n\to\infty} \int_0^{\pi} g_+(\theta) \sin((n+\tfrac{1}{2})\theta)\, d\theta = 0.$$

The result

$$\lim_{n\to\infty} \int_{-\pi}^{0} g_-(\theta) \sin((n+\tfrac{1}{2})\theta)\, d\theta = 0$$

is proved similarly.

Sec. 5.5, pp. 139–140.

#9 Fill in the missing steps in the proof of uniform convergence.

" Similarly $B_n = \frac{1}{n} A_n'$.   (16) "

Verification: $B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$

Parts: $\begin{cases} U = f(x) & dV = \sin(nx)dx \\ dU = f'(x)dx & V = -\frac{\cos(nx)}{n} \end{cases}$

$$= \frac{-1}{n\pi} f(x)\cos(nx)\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\cos(nx)\,dx$$

$$= \frac{-1}{n\pi}\left[f(\pi)(-1)^n - f(-\pi)(-1)^n \right] + \frac{1}{n} A_n'.$$

Since f is 2π-periodic, $f(\pi) = f(-\pi)$. Thus

$$B_n = \frac{1}{n} A_n', \quad (n = 1, 2, 3, \ldots).$$

" On the other hand, we know from Bessel's inequality [for the derivative function f'] that the infinite series

$$\sum_{n=1}^{\infty} \left(|A_n'|^2 + |B_n'|^2 \right) < \infty. \text{ "}$$

Verification: If $\Phi_n(x) = \cos(nx)$, $(n = 0, 1, 2, \ldots)$ and $\Psi_n(x) = \sin(nx)$, $(n = 1, 2, 3, \ldots)$ then the collection $\{\Phi_n\}_{n=0}^{\infty} \cup \{\Psi_n\}_{n=1}^{\infty}$ is an orthogonal set of functions on $(-\pi, \pi)$. (See pp. 103–104.)

If we apply Bessel's inequality (18), p. 128, to the derivative function f' on $(a,b) = (-\pi, \pi)$, then

$$(*) \quad \sum_{n=1}^{\infty} \left(|A_n'|^2 \int_{-\pi}^{\pi} |\Phi_n(x)|^2 dx + |B_n'|^2 \int_{-\pi}^{\pi} |\Psi_n(x)|^2 dx \right) \leq \int_{-\pi}^{\pi} |f'(x)|^2 dx < \infty$$

Sec. 5.5, pp. 139–140.

#9 (cont.) where the coefficients A_n' and B_n' are given by

$$A_n' = \frac{(f', \cos(nx))}{(\cos(nx), \cos(nx))} = \frac{\int_{-\pi}^{\pi} f'(x)\cos(nx)dx}{\int_{-\pi}^{\pi} \cos^2(nx)dx} = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos(nx)dx,$$

$$B_n' = \frac{(f', \sin(nx))}{(\sin(nx), \sin(nx))} = \frac{\int_{-\pi}^{\pi} f'(x)\sin(nx)dx}{\int_{-\pi}^{\pi} \sin^2(nx)dx} = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)dx,$$

$(n = 1, 2, 3, \ldots)$ and

$$\int_{-\pi}^{\pi} |X_n(x)|^2 dx = \int_{-\pi}^{\pi} \cos^2(nx)dx = \pi,$$

$$\int_{-\pi}^{\pi} |Y_n(x)|^2 dx = \int_{-\pi}^{\pi} \sin^2(nx)dx = \pi,$$

$(n = 1, 2, 3, \ldots)$. Substituting these expressions into Bessel's inequality $(*)$ yields

$$\sum_{n=1}^{\infty} \left(|A_n'|^2 \pi + |B_n'|^2 \pi\right) < \infty$$

and thus, the desired conclusion:

$$\sum_{n=1}^{\infty} \left(|A_n'|^2 + |B_n'|^2\right) < \infty .$$

"Therefore, ...

$$\ldots \sum_{n=1}^{\infty} \frac{1}{n}\left(|B_n'| + |A_n'|\right) \le \sqrt{2}\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2} \cdot \left(\sum_{n=1}^{\infty} \left(|A_n'|^2 + |B_n'|^2\right)\right)^{1/2} < \infty .$$

Here we have used Schwarz's inequality (Exercise 5, p. 139)."

#9 (cont.)

<u>Verification</u>: In this part of the problem we will need the following fact: For all real numbers x and y, $2(x^2+y^2) \geq (x+y)^2$.

Proof of fact:

$$
\begin{aligned}
2(x^2+y^2)-(x+y)^2 &= 2x^2+2y^2-x^2-2xy-y^2 \\
&= x^2-2xy+y^2 \\
&= (x-y)^2 \\
&\geq 0 .
\end{aligned}
$$

We apply the Schwarz inequality for infinite series

$$\sum_{n=1}^{\infty} a_n b_n \leq \left(\sum_{n=1} a_n^2\right)^{1/2} \cdot \left(\sum_{n=1}^{\infty} b_n^2\right)^{1/2}$$

(see exercise 5, p. 139) with $a_n = \frac{1}{n}$ and $b_n = |B_n'| + |A_n'|$, $n = 1, 2, 3, \ldots$ Therefore

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n}\left(|B_n'| + |A_n'|\right) &\leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2} \left[\sum_{n=1}^{\infty} \left(\overbrace{|B_n'|}^{x} + \overbrace{|A_n'|}^{y}\right)^2\right]^{1/2} \\
&\overset{\text{(fact above)}}{\leq} \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2} \cdot \left[\sum_{n=1}^{\infty} 2\left(\overbrace{|B_n'|^2}^{x^2} + \overbrace{|A_n'|^2}^{y^2}\right)\right]^{1/2} \\
&= \sqrt{2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2} \cdot \left[\sum_{n=1}^{\infty} \left(|B_n'|^2 + |A_n'|^2\right)\right]^{1/2} \\
&< \infty .
\end{aligned}
$$

Sec. 5.5, pp. 139-140.

#9 (cont.) "... we can write

$$\max_{-\infty<x<\infty} \left| f(x) - S_N(f;x) \right| \le \max_{-\infty<x<\infty} \sum_{n=N+1}^{\infty} \left| A_n \cos(nx) + B_n \sin(nx) \right|$$

Verification. By Theorem 4∞ of Section 5.4, the classical full Fourier series of f at x converges to $f(x)$ pointwise for all $x \in (-\infty, \infty)$. That is,

$$\begin{aligned} f(x) &= \lim_{N\to\infty} S_N(f;x) \\ &= \lim_{N\to\infty} \left\{ \frac{A_0}{2} + \sum_{n=1}^{N} \left[A_n \cos(nx) + B_n \sin(nx) \right] \right\} \\ &= \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos(nx) + B_n \sin(nx) \right] \end{aligned}$$

for all $x \in (-\infty, \infty)$. Therefore

$$\begin{aligned} \left| f(x) - S_N(f;x) \right| &= \Bigg| \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos(nx) + B_n \sin(nx)) - \\ &\qquad \left\{ \frac{A_0}{2} + \sum_{n=1}^{N} A_n \cos(nx) + B_n \sin(nx) \right\} \Bigg| \\ &= \left| \sum_{n=N+1}^{\infty} \left[A_n \cos(nx) + B_n \sin(nx) \right] \right| \end{aligned}$$

$$\Rightarrow \quad \left| f(x) - S_N(f;x) \right| \le \sum_{n=N+1}^{\infty} \left| A_n \cos(nx) + B_n \sin(nx) \right|$$

Taking the maximum of both members of the above inequality as x ranges over $(-\infty, \infty)$ yields the desired result.

Sec. 5.5, pp. 139-140.

#11 Prove that the full Fourier series of the function $f(x)=|x|$ in the interval $[-\pi,\pi]$ converges uniformly to f in $[-\pi,\pi]$.

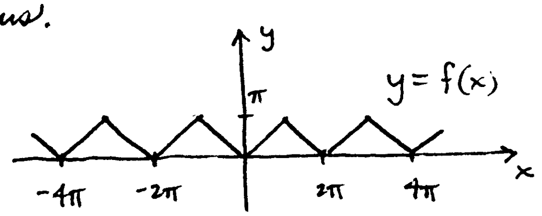
By exercise #14, p.114 (Sec. 5.2) with $l=\pi$, the full classical Fourier series of $f(x)=|x|$ on $(-\pi,\pi)$ is

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}\cos((2k-1)x).$$

Since $f(x)=|x|$ is continuous on $[-\pi,\pi]$, $f(-\pi)=f(\pi)$, and

$$f'(x) = \begin{cases} -1 & \text{if} \quad x\in(-\pi,0), \\ 1 & \text{if} \quad x\in(0,\pi), \end{cases}$$

is piecewise continuous on $(-\pi,\pi)$, the function f has a continuous 2π-periodic extension to the whole real line such that f' is piecewise continuous.



By Theorem 4$^\infty$ (Sec 5.4, p. 125) the Fourier series of f at x converges to $f(x)$ for all real x. In particular,

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos((2k-1)x) \qquad \text{for all } x\in[-\pi,\pi].$$

Thus $\max\limits_{-\pi\le x\le\pi} \left|\, |x| - S_{2N}(f;x)\right| = \max\limits_{-\pi\le x\le \pi} \left| |x| - S_{2N-1}(f;x)\right| =$

Sec. 5.5, pp. 139-140.

#11 (cont.)

$$= \max_{-\pi\le x\le\pi} \left| \overbrace{\frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}\cos((2k-1)x)}^{|x|} - \overbrace{\left(\frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{N} \frac{1}{(2k-1)^2}\cos((2k-1)x) \right)}^{S_{2N-1}(f;x)} \right|$$

$$= \max_{-\pi\le x\le\pi} \left| \frac{4}{\pi}\sum_{k=N+1}^{\infty} \frac{1}{(2k-1)^2}\cos((2k-1)x) \right|$$

$$\le \max_{-\pi\le x\le\pi} \frac{4}{\pi}\sum_{k=N+1}^{\infty} \left| \frac{1}{(2k-1)^2}\cos((2k-1)x) \right|$$

$$= \frac{4}{\pi}\sum_{k=N+1}^{\infty} \frac{1}{(2k-1)^2} .$$

The last expression is the tail of a convergent numerical series, and thus goes to zero as N tends to infinity. Therefore

$$\max_{-\pi\le x\le\pi} \left| |x| - S_M(f;x) \right| \to 0 \quad \text{as } M\to\infty;$$

i.e. the full Fourier series of $|x|$ in $[-\pi,\pi]$ converges uniformly to $|x|$ in $[-\pi,\pi]$.

Sec. 5.5, pp. 139–140.

#13. A very slick proof of the pointwise convergence of Fourier series due to P. Chernoff (American Mathematical Monthly, May 1980), goes as follows.

(a) Let f be a C^1 function of period 2π. First show that we may as well assume that $f(0)=0$ and we need only show that the Fourier series converges to zero at $x=0$.

(b) Let $g(x) = f(x)/(e^{ix}-1)$. Show that g is a continuous function.

(c) Let $\{C_n\}$ be the complex Fourier coefficients of f and $\{D_n\}$ the coefficients of g. Show that $D_n \to 0$ (as $|n| \to \infty$).

(d) Show that $C_n = D_{n-1} - D_n$ so that the series $\sum C_n$ is telescoping.

(e) Deduce that the Fourier series of f at $x=0$ converges to zero.
