#2. Prove the Schwarz inequality (for any pair of square-integrable functions):
\[(f, g) \leq \|f\| \cdot \|g\|.
\]
If \(g = 0\) then clearly equality holds in \((*)\):
\[|(f, g)| = 0 = \|f\| \cdot \|g\|.
\]
Therefore we may suppose that \(g \neq 0\), and consequently that
\[\|g\|^2 = (g, g) = \int_a^b [g(t)]^2 dt > 0.
\]
Consider the quadratic function of \(t\) given by
\[H(t) = \|f - tg\|^2 = (f - tg, f - tg)
\]
\[= (f, f) - 2t(f, g) + t^2(g, g).
\]
Then \(0 = H(t) = \frac{(f, g)^2}{(g, g)} - (g, g)t + t^2(g, g)
\]
implies \(t = \frac{(f, g)}{(g, g)}\). Since \(H''(t) = 2(g, g) > 0\) for all \(t\), the critical number \(t = (f, g)/(g, g)\) minimizes \(H:\)
\[0 \leq H\left(\frac{(f, g)}{(g, g)}\right) = (f, f) - 2 \frac{(f, g)^2}{(g, g)}(f, g) + \frac{(f, g)^2}{(g, g)}(g, g) = (f, f) - \frac{(f, g)^2}{(g, g)}
\]
so rearranging we have \((f, g)^2 \leq (f, f) \cdot (g, g)\). Extracting square roots of both sides of this inequality gives \((*)\):
\[|(f, g)| = \sqrt{(f, g)^2} \leq \sqrt{(f, f) \cdot (g, g)} = \|f\| \cdot \|g\|.
\]
Sec. 5.5, pp. 139–140.

4. (a) Solve the problem

\[ u_t - ku_{xx} = 0 \quad \text{for} \quad 0 < x < l, \quad 0 < t < \infty, \]

\[ u(x,0) = f(x) \quad \text{for} \quad 0 \leq x \leq l, \]

with the unusual boundary conditions

\[ u_x(0,t) = u_x(l,t) = \frac{u(l,t) - u(0,t)}{l} \quad \text{for} \quad 0 \leq t < \infty. \]

Assume that there are no negative eigenvalues.

(b) Assuming that you can take limits term by term (in the infinite series resulting from part (a)) show that

\[ \lim_{t \to \infty} u(x,t) = A + Bx. \]

(c) Use Green's first identity and exercise #3 to show that there are no negative eigenvalues.

(d) Find A and B.

(a) If we separate variables via \( u(x,t) = X(x)T(t) \) in this problem, we arrive at the coupled system

\[
\left\{ \begin{array}{l}
T' + \lambda k T = 0, \\
X'' + \lambda X = 0, \quad X(0) = X(l) = \frac{X(l) - X(0)}{l}.
\end{array} \right.
\]

It is easy to verify that \( \lambda = 0 \) is an eigenvalue and any (nonzero) linear function \( X(x) = Ax + B \) is a corresponding eigenfunction.

Suppose \( \lambda = \beta^2 > 0 \). The general solution to the ODE is \( X(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x) \).

Applying the B.C.'s gives

\[ 0 = X'(0) - X'(l) = \beta c_2 - \beta c_2 \cos(\beta l) + \beta c_2 \sin(\beta l) \]

\[ 0 = X'(0) - \frac{X(l) - X(0)}{l} = \beta c_2 - \frac{c_2 \cos(\beta l) + c_2 \sin(\beta l) - c_2}{l} \]

The existence of a nontrivial solution to this linear homogeneous system of
(a) (cont.) equations in the variables $c$ and $c_1$ is equivalent to the vanishing of the determinant of the coefficient matrix:

$$
\begin{align*}
0 &= (1 - \cos(\beta l)) c_2 + \sin(\beta l) c_1 \\
0 &= (\beta l - \sin(\beta l)) c_2 + (1 - \cos(\beta l)) c_1 \\
0 &= \begin{vmatrix} \sin(\beta l) & 1 - \cos(\beta l) \\ 1 - \cos(\beta l) & \beta l - \sin(\beta l) \end{vmatrix} = (\beta l \sin(\beta l) - \sin^2(\beta l)) - 1 + 2 \cos(\beta l) - \cos^2(\beta l)
\end{align*}
$$

$$
0 = \beta l \sin(\beta l) - 2 + 2 \cos(\beta l) \quad \Rightarrow \quad \beta l \sin(\beta l) = 2(1 - \cos(\beta l))
$$

$$
\Rightarrow \begin{cases} 
\beta l = 2n\pi, & (n = 1, 2, 3, \ldots) \\
\text{or} \\
\frac{\beta l}{2} = \frac{1 - \cos(\beta l)}{\sin(\beta l)} = \frac{\tan(\beta l)}{1} 
\end{cases}
$$

In the first case we have

$$
\beta_n = \frac{3 \pi}{l} \quad (n = 1, 2, 3, \ldots)
$$

and in the second case

$$
\beta_n = \frac{2n\pi}{l}, \quad (2n+1)\frac{\pi}{l} \quad (n = 1, 2, 3, \ldots)
$$

with

$$
\lim_{n \to \infty} \left[ \beta_n - (2n+1)\frac{\pi}{l} \right] = 0.
$$

(Up to a constant multiple)

In the first case, the corresponding eigenfunctions are

$$X_n(x) = \cos(\beta_n x) = \cos\left(\frac{3 \pi x}{l}\right), \quad (n = 1, 2, 3, \ldots)$$

and in the second case they are (up to a constant multiple)

$$\tilde{X}_n(x) = \tilde{\beta}_n \cos(\tilde{\beta}_n x) - 2\sin(\tilde{\beta}_n x), \quad (n = 1, 2, 3, \ldots).$$

Thus a (formal) solution is

$$u(x,t) = A + B + \sum_{n=1}^{\infty} c_n e^{-(\beta_n^2 k^2 t)} \cos(\beta_n x) + \sum_{n=1}^{\infty} \tilde{c}_n e^{-(\tilde{\beta}_n^2 k^2 t)} \left[ \tilde{\beta}_n \cos(\tilde{\beta}_n x) - 2\sin(\tilde{\beta}_n x) \right].$$
#4(a) (cont.) where the coefficients \( c_n \) and \( \tilde{c}_n \) \( (n=1,2,3,\ldots) \) are chosen so that

\[
q(x) = u(x,0) = Ax + B + \sum_{n=1}^{\infty} c_n \cos(\beta_n x) + \sum_{n=1}^{\infty} \tilde{c}_n \left( \frac{\beta_n}{\beta_n^2 + \lambda^2} \right) \left[ \cos(\beta_n x) - 2 \sin(\beta_n x) \right]
\]

for all \( 0 \leq x \leq l \).

(b) Taking the formal limit (i.e. term by term in the infinite series expression for \( u = u(x,t) \)), we have

\[
\lim_{t \to \infty} u(x,t) = \lim_{t \to \infty} \left\{ Ax + B + \sum_{n=1}^{\infty} c_n \cos(\beta_n x) + \sum_{n=1}^{\infty} \tilde{c}_n \left( \frac{\beta_n}{\beta_n^2 + \lambda^2} \right) \left[ \cos(\beta_n x) - 2 \sin(\beta_n x) \right] \right\} = Ax + B.
\]

(c) Let \( \lambda = -\beta^2 < 0 \) and let \( \Xi = \Xi(x) \) be a solution to

\[
(\star) \quad \left\{ \begin{array}{l}
\Xi'' + \lambda \Xi = \Xi'' - \beta^2 \Xi = 0 \quad \text{for} \quad 0 < x < l,

(\star \star) \quad \Xi'(0) = \Xi'(l) = \frac{\Xi(l) - \Xi(0)}{l}.
\end{array} \right.
\]

By Green's first identity (exercise #12, Sec. 5.3) or integration by parts

\[
(\star \star \star) \quad \int_0^l \Xi''(x) \Xi(x) \, dx = - \int_0^l \left[ \Xi'(x) \right]^2 \, dx + \left. \Xi(x) \Xi(x) \right|_0^l.
\]

But \( \int_0^l \Xi''(x) \Xi(x) \, dx = \beta^2 \int_0^l \left[ \Xi(x) \right]^2 \, dx \) by \((\star)\),

\[-\int_0^l \left[ \Xi'(x) \right]^2 \, dx \leq - \frac{\left[ \Xi(l) - \Xi(0) \right]^2}{l} \) by exercise #3,

and

\[
\left. \Xi(x) \Xi(x) \right|_0^l = \Xi'(l) \Xi(l) - \Xi'(0) \Xi(0) = \left( \Xi(l) - \Xi(0) \right) \left( \Xi(l) - \Xi(0) \right) \) by \((\star \star)\).
Sec. 5.5, pp. 139-140.

#4 (c) (cont.) Substituting these expressions into (***) gives

\[ \beta^2 \int_0^l [\bar{X}(x)]^2 \, dx \leq \frac{[\bar{X}(\ell) - \bar{X}(0)]^2}{\lambda} + \frac{[\bar{X}(\ell) - \bar{X}(0)]^2}{\ell} \leq 0. \]

Dividing by the positive quantity \( \beta^2 \) we have

\[ 0 \leq \int_0^l [\bar{X}(x)]^2 \, dx \leq 0 \quad \text{and hence} \quad \int_0^l [\bar{X}(x)]^2 \, dx = 0. \]

Clearly, from above.

By continuity of \( \bar{X} \) on \((0, \ell)\) it then follows \( \bar{X}(x) = 0 \) for all \( 0 < x < \ell \),

i.e. \( \lambda = -\beta^2 < 0 \) is not an eigenvalue of 

\((*) - (***)\).

(d) By Theorem 1, Sec 5.3, the eigenfunctions \( Ax + B, \{ \cos(\beta_n x) \}_{n=1}^\infty \),

and \( \{ \beta_n \cos(\beta_n x) - 2 \sin(\beta_n x) \}_{n=1}^\infty \) of the eigenvalue problem \((*) - (***)\) in

part (c) are orthogonal on \((0, \ell)\). Therefore the constants in

the expansion

\((*) \quad \phi(x) = u(x, 0) = Ax + B + \sum_{n=1}^\infty c_n \cos(\beta_n x) + \sum_{n=1}^\infty \tilde{c}_n \left[ \beta_n \cos(\beta_n x) - 2 \sin(\beta_n x) \right] \)

of \( \phi \) on \((0, \ell)\) are the Fourier coefficients of \( \phi \) with respect to

the orthogonal set of eigenfunctions.

The two-dimensional space of eigenfunctions corresponding to

the eigenvalue \( \lambda = 0 \), \( \{ Ax + B : A, B \text{ are real numbers} \} \), has an

orthogonal basis \( 1 \) and \( x - \frac{\ell}{2} \) on \((0, \ell)\). Therefore, the particular

choice of constants \( A \) and \( B \) in the expansion \((*)\) of \( \phi \) satisfies

\((***) \quad Ax + B = A(x - \frac{\ell}{2}) + B \cdot 1 \quad \text{for} \quad 0 < x < \ell \)
Sec. 5.5, pp. 139-140.

#4 (d) (cont.) where $A$ and $B'$ are given by the familiar Fourier coefficient formulas:

$$A = \frac{\langle \varphi, x - \frac{L}{2} \rangle}{\langle x - \frac{L}{2}, x - \frac{L}{2} \rangle} \quad \text{and} \quad B' = \frac{\langle \varphi, \frac{1}{2} \rangle}{\langle 1, 1 \rangle}.$$ 

But $(1, 1) = \int_0^L 1^2 \, dx = L$ 

and 

$$(x - \frac{L}{2}, x - \frac{L}{2}) = \int_0^L (x - \frac{L}{2})^2 \, dx = \frac{1}{3} (x - \frac{L}{2})^3 \bigg|_0^L = \frac{1}{3} \left( \frac{L}{2} \right)^3 - \frac{1}{3} \left( -\frac{L}{2} \right)^3 = \frac{L^3}{12},$$ 

From (††) it is clear that 

$$B = B' - \frac{L}{2} A,$$ 

so substituting from (††) we have 

$$B = \frac{1}{\lambda^2} \int_0^L \varphi(x) \, dx - \frac{1}{\lambda^2} \int_0^L \varphi(x) \left( x - \frac{L}{2} \right) \, dx$$ 

and 

$$B = \frac{2}{\lambda^2} \int_0^L \varphi(x) \left( \frac{L}{2} \right) \, dx - \frac{2}{\lambda^2} \int_0^L \varphi(x) \left( 3x - \frac{3L}{2} \right) \, dx$$ 

Finally, 

$$B = \frac{2}{\lambda^2} \int_0^L \varphi(x) \left( 2x - 3x \right) \, dx.$$
Prove that both integrals in
\[\int_{0}^{\pi} g_+^2(\theta) \sin(n+\frac{1}{2})\theta \, d\theta + \int_{-\pi}^{0} g_-^2(\theta) \sin(n+\frac{1}{2})\theta \, d\theta\]
tend to zero (so \( N \) tends to infinity).

Because the quotient of two piecewise continuous functions is piecewise continuous as long as the denominator isn't zero, it follows that
\[g_+^2(\theta) = \frac{f(x+\theta) - f(x^+)}{\sin(\theta/2)}\]
is piecewise continuous for \( \theta \in (0, \pi) \). Also, l'Hôpital's rule implies
\[\lim_{\theta \to 0^+} g_+^2(\theta) = \lim_{\theta \to 0^+} \frac{f(x+\theta) - f(x^+)}{\sin(\theta/2)} = \lim_{\theta \to 0^+} \frac{f'(x+\theta)}{1/2 \cdot \cos(\theta/2)} = 2f'(x^+)\]

Therefore, the discontinuity in \( g_+ \) at \( \theta = 0 \) is removable, and consequently
\[\int_{0}^{\pi} |g_+^2(\theta)| \, d\theta < \infty\]. If we apply Bessel's inequality (18), p.128, with \( X_n(\theta) = \sin[(n+\frac{1}{2})\theta] \), \((a,b) = (0,\pi)\), we get

\[\sum_{n=1}^{\infty} A_n^2 \int_{0}^{\pi} |X_n(\theta)|^2 \, d\theta \leq \int_{0}^{\pi} |g_+^2(\theta)| \, d\theta\]

where
\[A_n = \frac{\langle g_+, X_n \rangle}{\langle X_n, X_n \rangle} = \frac{\int_{0}^{\pi} g_+(\theta) \sin(n+\frac{1}{2})\theta \, d\theta}{\int_{0}^{\pi} \sin^2(n+\frac{1}{2})\theta \, d\theta} = \frac{\pi}{\pi} \int_{0}^{\pi} g_+(\theta) \sin(n+\frac{1}{2})\theta \, d\theta\]
Sec. 5.5, pp. 139-140.
#8 (cont.) Substituting these expressions into Bessel's inequality (*) gives
\[
\sum_{n=1}^{\infty} \left\{ \frac{2}{\pi} \int_0^{\pi} g_+(\theta) \sin((n+\frac{1}{2})\theta) d\theta \right\}^2 \cdot \frac{\pi}{2} \leq \int_0^{\pi} |g_+(\theta)|^2 d\theta < \infty,
\]
and consequently
\[
\sum_{n=1}^{\infty} \left\{ \int_0^{\pi} g_+(\theta) \sin((n+\frac{1}{2})\theta) d\theta \right\}^2 < \infty.
\]

Since the terms in a convergent series must go to zero as \( n \to \infty \), we get the desired result:
\[
\lim_{n \to \infty} \left\{ \int_0^{\pi} g_+(\theta) \sin((n+\frac{1}{2})\theta) d\theta \right\}^2 = 0
\]
and hence
\[
\lim_{n \to \infty} \int_0^{\pi} g_+(\theta) \sin((n+\frac{1}{2})\theta) d\theta = 0.
\]

The result
\[
\lim_{n \to \infty} \int_{-\pi}^{0} g_-(\theta) \sin((n+\frac{1}{2})\theta) d\theta = 0
\]

is proved similarly.
9. Fill in the missing steps in the proof of uniform convergence.

Similarly, \( B_n = \frac{1}{n} A_n'. \) (16)

Verification:

\[
B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx
\]

\[
\begin{align*}
\text{Parts:} & \quad u = f(x) \quad dv = \sin(nx) \, dx \\
& \quad du = f'(x) \, dx \quad v = -\frac{\cos(nx)}{n}
\end{align*}
\]

\[
= \frac{-1}{n\pi} f(\pi) \cos(n\pi) + \frac{1}{n\pi} \int_{-\pi}^{\pi} f'(x) \cos(nx) \, dx
\]

\[
= \frac{-1}{n\pi} \left[ f(\pi)(-1)^n - f(-\pi)(-1)^n \right] + \frac{1}{n} A_n'.
\]

Since \( f \) is 2\( \pi \)-periodic, \( f(\pi) = f(-\pi) \). Thus

\[
B_n = \frac{1}{n} A_n', \quad (n=1, 2, 3, \ldots)
\]

On the other hand, we know from Bessel's inequality [for the derivative function \( f' \)] that the infinite series

\[
\sum_{n=1}^{\infty} (|A_n'|^2 + |B_n'|^2) < \infty.
\]

Verification: If \( \Xi_n(x) = \cos(nx), \quad (n=0, 1, 2, \ldots) \) and \( \Upsilon_n(x) = \sin(nx), \quad (n=1, 2, 3, \ldots) \) then the collection \( \{\Xi_n\}_{n=0}^{\infty} \cup \{\Upsilon_n\}_{n=1}^{\infty} \) is an orthogonal set of functions on \((-\pi, \pi)\). (See pp. 103-104.)

If we apply Bessel's inequality (18), p.128, to the derivative function \( f' \) on \((a,b) = (-\pi, \pi)\), then

\[
(*) \quad \sum_{n=1}^{\infty} \left( |A_n'|^2 \int_{-\pi}^{\pi} |\Xi_n(x)|^2 \, dx + |B_n'|^2 \int_{-\pi}^{\pi} |\Upsilon_n(x)|^2 \, dx \right) \leq \int_{-\pi}^{\pi} |f'(x)|^2 \, dx < \infty
\]
where the coefficients $A_n'$ and $B_n'$ are given by

$$
A_n' = \frac{f'(x) \cos(nx)}{\cos^n(x) \sin(nx)} = \frac{1}{n} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx,
$$

$$
B_n' = \frac{f'(x) \sin(nx)}{\sin^n(x) \sin(nx)} = \frac{1}{n} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx,
$$

$(n=1, 2, 3, \ldots)$ and

$$
\int_{-\pi}^{\pi} |\xi_n(x)|^2 \, dx = \int_{-\pi}^{\pi} \cos^2(nx) \, dx = \pi,
$$

$$
\int_{-\pi}^{\pi} |\xi_n(x)|^2 \, dx = \int_{-\pi}^{\pi} \sin^2(nx) \, dx = \pi,
$$

$(n=1, 2, 3, \ldots)$. Substituting these expressions into Bessel's inequality (*) yields

$$
\sum_{n=1}^{\infty} \left( |A_n'|^2 + |B_n'|^2 \right) < \infty
$$

and thus, the desired conclusion:

$$
\sum_{n=1}^{\infty} \left( |A_n|^2 + |B_n|^2 \right) < \infty.
$$

Therefore, ...

$$
\sum_{n=1}^{\infty} \frac{1}{n} \left( |B_n'| + |A_n'| \right) \leq \sqrt{\left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} \left( \sum_{n=1}^{\infty} (|A_n'|^2 + |B_n'|^2) \right)^{1/2}} < \infty.
$$

Here we have used Schwarz's inequality (Exercise 5, p.139).
Verification: In this part of the problem we will need the following fact: For all real numbers \( x \) and \( y \), \( 2(x^2+y^2) \geq (x+y)^2 \).

Proof of fact: \[
2(x^2+y^2)-(x+y)^2 = 2x^2+2y^2-x^2-2xy-y^2 \\
= x^2-2xy+y^2 \\
= (x-y)^2 \\
\geq 0.
\]

We apply the Schwarz inequality for infinite series
\[
\sum_{n=1}^{\infty} a_n b_n \leq \left( \sum_{n=1}^{\infty} a_n^2 \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} b_n^2 \right)^{\frac{1}{2}}
\]
(see exercise 5, p. 139) with \( a_n = \frac{1}{n} \) and \( b_n = |B_n|^1 + |A_n|^1 \),\( n = 1, 2, 3, \ldots \) Therefore
\[
\sum_{n=1}^{\infty} \frac{1}{n} (|B_n|^1 + |A_n|^1) \leq \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} (|B_n|^1 + |A_n|^1)^2 \right)^{\frac{1}{2}}
\]
(\text{fact above})
\[
\left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} 2(|B_n|^2 + |A_n|^2) \right)^{\frac{1}{2}}
\]
\[
= \sqrt{2} \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} \left[ \sum_{n=1}^{\infty} (|B_n|^2 + |A_n|^2) \right]^{\frac{1}{2}}
\]
\[
< \infty.
\]
Theorem 6.5, pp. 139-140.

(Cont.) "... we can write

\[
\max_{-\infty < x < \infty} |f(x) - S_N(f; x)| \leq \max_{-\infty < x < \infty} \sum_{n=N+1}^{\infty} |A_n \cos(nx) + B_n \sin(nx)|
\]

Verification. By Theorem 4.4 of Section 5.4, the classical Fourier series of \( f \) at \( x \) converges to \( f(x) \) pointwise for all \( x \in (-\infty, \infty) \). That is,

\[
f(x) = \lim_{N \to \infty} S_N(f; x)
\]

\[
= \lim_{N \to \infty} \left\{ \frac{A_0}{2} + \sum_{n=1}^{N} [A_n \cos(nx) + B_n \sin(nx)] \right\}
\]

\[
= \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos(nx) + B_n \sin(nx)]
\]

for all \( x \in (-\infty, \infty) \). Therefore

\[
|f(x) - S_N(f; x)| = \left| \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos(nx) + B_n \sin(nx)) - \left\{ \frac{A_0}{2} + \sum_{n=1}^{N} A_n \cos(nx) + B_n \sin(nx) \right\} \right|
\]

\[
= \left| \sum_{n=N+1}^{\infty} [A_n \cos(nx) + B_n \sin(nx)] \right|
\]

\[
\Rightarrow |f(x) - S_N(f; x)| \leq \sum_{n=N+1}^{\infty} |A_n \cos(nx) + B_n \sin(nx)|
\]

Taking the maximum of both members of the above inequality as \( x \) ranged over \( (-\infty, \infty) \) yields the desired result.
#11 Prove that the full Fourier series of the function $f(x) = |x|$ in the interval $[-\pi, \pi]$ converges uniformly to $f$ in $[-\pi, \pi]$.

By exercise #14, p.114 (Sec. 5.2) with $l=\pi$, the full classical Fourier series of $f(x) = |x|$ on $(-\pi, \pi)$ is

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos((2k-1)x).$$

Since $f(x) = |x|$ is continuous on $[-\pi, \pi]$, $f(-\pi) = f(\pi)$, and

$$f'(x) = \begin{cases} -1 & \text{if } x \in (-\pi, 0), \\ 1 & \text{if } x \in (0, \pi), \end{cases}$$

is piecewise continuous on $(-\pi, \pi)$, the function $f$ has a continuous $2\pi$-periodic extension to the whole real line such that $f'$ is piecewise continuous.

By Theorem 4 (Sec 5.4, p. 125) the Fourier series of $f(x)$ converges to $f(x)$ for all real $x$. In particular,

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos((2k-1)x) \quad \text{for all } x \in [-\pi, \pi].$$

Thus

$$\max_{-\pi \leq x \leq \pi} \left| |x| - S_N(f;x) \right| = \max_{-\pi \leq x \leq \pi} \left| |x| - S_{2N-1}(f;x) \right| =$$
\[ \| f \|_1 = \text{max} \left| \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos((2k-1)x) \right| \]

\[ \leq \text{max} \left| \frac{4}{\pi} \sum_{k=N+1}^{\infty} \frac{1}{(2k-1)^2} \right| \]

\[ = \frac{4}{\pi} \sum_{k=N+1}^{\infty} \frac{1}{(2k-1)^2}. \]

The last expression is the tail of a convergent numerical series, and thus goes to zero as \( N \) tends to infinity. Therefore

\[ \text{max} \left| 1 \times 1 - \sum_{1}^{M} f_j(x) \right| \to 0 \quad \text{as} \quad M \to \infty; \]

i.e., the full Fourier series of \( 1 \times 1 \) in \([\pi, \pi]\) converges uniformly to \( 1 \times 1 \) in \([\pi, \pi]\).
#13. A very slick proof of the pointwise convergence of Fourier series due to P. Chernoff (American Mathematical Monthly, May 1980), goes as follows.

(a) Let $f$ be a $C^1$ function of period $2\pi$. First show that we may as well assume that $f(0) = 0$ and we need only show that the Fourier series converges to zero at $x = 0$.

(b) Let $g(x) = f(x)/(e^{ix} - 1)$. Show that $g$ is a continuous function.

(c) Let $\{C_n\}$ be the complex Fourier coefficients of $f$ and $\{D_n\}$ the coefficients of $g$. Show that $D_n \to 0$ as $|n| \to \infty$.

(d) Show that $C_n = D_{n-1} - D_n$ so that the series $\sum C_n$ is telescoping.

(e) Deduce that the Fourier series of $f$ at $x = 0$ converges to zero.