Sec. 5.6, pp. 144-145.

#1. (a) Solve as a series the equation $u_t - u_{xx} = 0$ for $0 < x < 1$, $0 < t < \infty$, with $u_x(0,t) = u(1,t) - 1 = 0$ for $0 \le t < \infty$ and $u(x,0) = x^2$ for $0 \le x \le 1$. Compute the first two coefficients explicitly.

(b) What is the equilibrium state (the term that does not tend to zero)?

We use the method of shifting the data.

Let $v(x,t) = u(x,t) - 1$

where $u = u(x,t)$ is a solution to the problem in part (a). Then v satisfies

$$(*) \quad \begin{cases} v_t - v_{xx} = 0 & \text{for } 0 < x < 1,\ 0 < t < \infty, \\ v_x(0,t) = v(1,t) = 0 & \text{for } t \ge 0, \\ v(x,0) = x^2 - 1 & \text{for } 0 \le x \le 1. \end{cases}$$

We solve (*) by separation of variables. Let $v(x,t) = X(x)T(t)$. Substituting in the PDE, separating variables, and using the homogeneous B.C.'s leads to

$$(**) \quad \begin{cases} T' + \lambda T = 0, \\ X'' + \lambda X = 0, \quad X'(0) = X(1) = 0. \end{cases}$$

By theorems 2 and 3 of Sec. 5.3, the eigenvalues of the problem in the second line of (**) are real and nonnegative. Explicitly we find that the eigenvalues are

$$\lambda_n = \beta_n^2 = \left(n + \tfrac{1}{2}\right)^2 \pi^2, \quad (n = 0, 1, 2, \ldots)$$

#1. (a) (cont.) and the corresponding eigenfunctions are

$$X_n(x) = \cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right), \quad (n=0,1,2,\dots).$$

The solutions to the first line of (**) with $\lambda = \lambda_n$ are, up to a constant multiple,

$$T_n(t) = e^{-(n+\frac{1}{2})^2\pi^2 t}, \quad (n=0,1,2,\dots).$$

Formally, then,

$$v(x,t) = \sum_{n=0}^{\infty} c_n e^{-(n+\frac{1}{2})^2\pi^2 t} \cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right).$$

The constants $c_0, c_1, c_2, \dots$ are chosen so as to satisfy the nonhomogeneous B.C. in (*):

$$x^2-1 = v(x,0) = \sum_{n=0}^{\infty} c_n \cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right) \quad \text{for } 0 \le x \le 1.$$

Thus the c_n's are the Fourier coefficients of x^2-1 with respect to the orthogonal set $\left\{\cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right)\right\}_{n=0}^{\infty}$ on $(0,1)$:

$$c_n = \frac{\left(x^2-1,\ \cos\left(\left(n+\frac{1}{2}\right)\pi x\right)\right)}{\left(\cos\left(\left(n+\frac{1}{2}\right)\pi x\right),\ \cos\left(\left(n+\frac{1}{2}\right)\pi x\right)\right)} = 2\int_0^1 (x^2-1)\cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right)dx.$$

Integrating by parts twice we find

$$c_n = \frac{-4}{\left[\left(n+\frac{1}{2}\right)\pi\right]^3}\sin\left(\left(n+\tfrac{1}{2}\right)\pi\right) = \frac{4(-1)^{n+1}}{\left[\left(n+\frac{1}{2}\right)\pi\right]^3}.$$

Thus, by theorem 2 of Sec. 5.4,

$$x^2-1 = \sum_{n=0}^{\infty} \frac{4(-1)^{n+1}}{\left[\left(n+\frac{1}{2}\right)\pi\right]^3}\cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right) \quad \text{for } 0 \le x \le 1,$$

Sec. 5.6, pp. 144-145.

#1 (a) (cont.) and the convergence is uniform. Therefore

$$\boxed{u(x,t) = 1 + v(x,t) = 1 + \sum_{n=0}^{\infty} \frac{4(-1)^{n+1}}{\left[(n+\frac{1}{2})\pi\right]^3} \cos\left((n+\tfrac{1}{2})\pi x\right) e^{-(n+\frac{1}{2})^2\pi^2 t}}$$

for $0 \le x \le 1$, $0 \le t < \infty$.

(b) The equilibrium state is the (constant) function

$$\boxed{u(x) = \lim_{t \to +\infty} u(x,t) = 1}.$$

Sec. 5.6, pp. 144-145

#3 (a) Solve as a series the equation $u_t - u_{xx} = 0$ for $0<x<1$, $0<t<\infty$, with $u_x(0,t) = 0$, $u_x(1,t) = 1$ for $t \geq 0$ and $u(x,0) = x^2$ for $0 \leq x \leq 1$. Compute the first two coefficients explicitly.

(b) What is the equilibrium state (the term that does not tend to zero)?

(a) We use a combination of the methods of subtraction and expansion.

Let $v(x,t) = u(x,t) - x^2/2$. This implies that $v_t = u_t$ and $v_{xx} = u_{xx} - 1$.

Also $v_x(x,t) = u_x(x,t) - x$ so

$$v_x(0,t) = u_x(0,t) - 0 = 0,$$

$$v_x(1,t) = u_x(1,t) - 1 = 0.$$

Finally $v(x,0) = u(x,0) - x^2/2 = x^2 - x^2/2 = x^2/2$. Thus the problem in part (a) is transformed into:

$$(\dagger)\quad \begin{cases} v_t - v_{xx} = 1 & \text{for } 0<x<1,\ 0<t<\infty, \\ v_x(0,t) = v_x(1,t) = 0 & \text{for } t \geq 0, \\ v_x(x,0) = x^2/2 & \text{for } 0 \leq x \leq 1. \end{cases}$$

The corresponding homogeneous problem has eigenfunctions $X_n(x) = \cos(n\pi x)$, $n = 0, 1, 2, \ldots$ Since this family is a complete orthogonal set of functions on $(0,1)$, for each $t \geq 0$ we have the expansion

$$(1)\qquad v(x,t) = \frac{1}{2} v_0(t) + \sum_{n=1}^{\infty} v_n(t)\cos(n\pi x) \quad \text{where } v_n(t) = 2\int_0^1 v(x,t)\cos(n\pi x)\,dx. \qquad (1')$$

Similarly,

Sec. 5.6, pp. 144-145

#3a (cont.)

(2) $$\frac{\partial v}{\partial t}(x,t) = \frac{1}{2}w_0(t) + \sum_{n=1}^{\infty} w_n(t)\cos(n\pi x) \quad \text{where} \quad w_n(t) = 2\int_0^1 v_t(x,t)\cos(n\pi x)\,dx, \qquad (2')$$

(3) $$\frac{\partial^2 v}{\partial x^2}(x,t) = \frac{1}{2}y_0(t) + \sum_{n=1}^{\infty} y_n(t)\cos(n\pi x) \quad \text{where} \quad y_n(t) = 2\int_0^1 v_{xx}(x,t)\cos(n\pi x)\,dx. \qquad (3')$$

From (†), (2'), and (3'), it follows that

$$w_n(t) - y_n(t) = 2\int_0^1 \left[v_t(x,t) - v_{xx}(x,t)\right]\cos(n\pi x)\,dx$$

$$= 2\int_0^1 1\cdot\cos(n\pi x)\,dx$$

$$= \begin{cases} \left.\dfrac{2\sin(n\pi x)}{n\pi}\right|_0^1 & \text{if } n \neq 0, \\ 2 & \text{if } n = 0. \end{cases}$$

Thus

$$(*) \quad w_n(t) - y_n(t) = \begin{cases} 0 & \text{if } n \neq 0, \\ 2 & \text{if } n = 0. \end{cases}$$

From (1') and (2'), we have

$$(**) \quad w_n(t) = \frac{d}{dt}\left(2\int_0^1 v(x,t)\cos(n\pi x)\,dx\right) = v_n'(t).$$

Integrating (3') by parts twice (or applying Green's second identity (3), p. 115 to (3')) gives:

$$y_n(t) = 2\left[\cos(n\pi x)v_x(x,t) + n\pi\sin(n\pi x)v(x,t)\right]\Bigg|_{x=0}^{1} - 2(n\pi)^2\int_0^1 v(x,t)\cos(n\pi x)\,dx.$$

Sec. 5.6, pp. 144–145.

#3(a)(cont.) Making use of the boundary conditions from (†): $v_x(0,t) = v_x(1,t) = 0$, we see that the boundary terms in the previous expression for $y_n(t)$ vanish, and hence by (1′):

$$(***) \qquad y_n(t) = -(n\pi)^2 v_n(t).$$

Substituting from (**) and (***) into (*) yields

$$v_n'(t) + (n\pi)^2 v_n(t) = \begin{cases} 2 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

From (1′) and the initial condition $v(x,0) = x^2/2$ in (†), we see that

$$v_n(0) = 2\int_0^1 v(x,0)\cos(n\pi x)\,dx = \int_0^1 x^2 \cos(n\pi x)\,dx.$$

Integrating by parts twice produces

$$v_n(0) = \begin{cases} \left.\dfrac{x^2 \sin(n\pi x)}{n\pi} + \dfrac{2x\cos(n\pi x)}{(n\pi)^2} - \dfrac{2\sin(n\pi x)}{(n\pi)^3}\right|_{x=0}^{1} & \text{if } n \neq 0, \\ \left. x^3/3 \right|_{x=0}^{1} & \text{if } n = 0, \end{cases}$$

$$= \begin{cases} \dfrac{2(-1)^n}{(n\pi)^2} & \text{if } n \neq 0, \\ \dfrac{1}{3} & \text{if } n = 0. \end{cases}$$

<u>$n=0$</u>: $v_0'(t) = 2$ and $v_0(0) = 1/3$ implies $v_0(t) = 2t - 1/3$.

Integrating factor

<u>$n \neq 0$</u>: $v_n'(t) + (n\pi)^2 v_n(t) = 0$ implies $e^{(n\pi)^2 t} v_n'(t) + (n\pi)^2 e^{(n\pi)^2 t} v_n(t) = 0$

and $\dfrac{d}{dt}\left[e^{(n\pi)^2 t} v_n(t)\right] = 0.$

Sec. 5.6, pp. 144–145.

#3(a)(cont.) Integrating both sides gives

$$e^{(n\pi)^2 t} v_n(t) = \text{constant},$$

$$v_n(t) = (\text{constant}) e^{-(n\pi)^2 t}.$$

$$v_n(0) = \frac{2(-1)^n}{(n\pi)^2} \quad \text{so} \quad v_n(t) = \frac{2(-1)^n}{(n\pi)^2} e^{-(n\pi)^2 t}, \qquad (n = 1, 2, 3, \ldots).$$

Therefore $v(x,t) = \frac{1}{2} v_0(t) + \sum_{n=1}^{\infty} v_n(t) \cos(n\pi x)$

$$= t - 1/6 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{(n\pi)^2} e^{-(n\pi)^2 t} \cos(n\pi x).$$

Since $v(x,t) = u(x,t) - x^2/2$, it follows that

$$u(x,t) = \frac{x^2}{2} + t - \frac{1}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{(n\pi)^2} e^{-(n\pi)^2 t} \cos(n\pi x)$$

$$= \frac{x^2}{2} + t - \frac{1}{6} + \frac{2}{\pi^2}\left(- e^{-\pi^2 t} \cos(\pi x) + \frac{1}{4} e^{-4\pi^2 t} \cos(2\pi x) - \ldots \right)$$

(b) The term in $u(x,t)$ that does not tend to zero as $t \to \infty$ is

$$\boxed{\frac{x^2}{2} + t - \frac{1}{6}}.$$

Note that this term satisfies the PDE $u_t - u_{xx} = 0$ and the boundary conditions $u_x(0,t) = 0$ and $u_x(1,t) = 1$ for $t \geq 0$. It is the "steady state" solution.

Sec. 5.6, pp. 144-145.

#5. Solve:

$$(\dagger)\quad \begin{cases} u_{tt} - c^2 u_{xx} = e^t \sin(5x) & \text{for } 0<x<\pi,\ -\infty<t<\infty, \\ u(0,t) = u(\pi,t) = 0 & \text{for } -\infty<t<\infty, \\ u(x,0) = 0 \text{ and } u_t(x,0) = \sin(3x) & \text{for } 0\le x\le\pi. \end{cases}$$

The eigenfunctions for the associated homogeneous problem are $X_n(x) = \sin(nx)$, $(n=1,2,3,\ldots)$. Hence for each fixed $t\in(-\infty,\infty)$ and all $x\in(0,\pi)$,

(1) $\quad u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin(nx)$ where $u_n(t) = \frac{2}{\pi}\int_0^\pi u(x,t)\sin(nx)\,dx$. (1′)

(1a) $\quad u_t(x,t) = \sum y_n(t)\sin(nx)$ where $y_n(t) = \frac{2}{\pi}\int_0^\pi u_t(x,t)\sin(nx)\,dx$ (1a′)

(2) $\quad u_{tt}(x,t) = \sum_{n=1}^{\infty} v_n(t)\sin(nx)$ where $v_n(t) = \frac{2}{\pi}\int_0^\pi u_{tt}(x,t)\sin(nx)\,dx$ (2′)

(3) $\quad u_{xx}(x,t) = \sum_{n=1}^{\infty} w_n(t)\sin(nx)$ " $w_n(t) = \frac{2}{\pi}\int_0^\pi u_{xx}(x,t)\sin(nx)\,dx$ (3′)

From (2′) and (1′), $\quad v_n(t) = \frac{d^2}{dt^2}\left(\frac{2}{\pi}\int_0^\pi u(x,t)\sin(nx)\,dx\right) = u_n''(t)$. $\quad (*)$

From (2′), (3′), and $(\dagger)$

$$(**)\qquad v_n(t) - c^2 w_n(t) = \frac{2}{\pi}\int_0^\pi \left[u_{tt}(x,t) - c^2 u_{xx}(x,t)\right]\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^\pi e^t \sin(5x)\sin(nx)\,dx$$

$$= \begin{cases} 0 & \text{if } n\ne 5, \\ e^t & \text{if } n=5. \end{cases}$$

From (3′), it follows by integrating by parts twice that

$$\begin{cases} v = \sin(nx) & dV = u_{xx}(x,t)\,dx \\ dv = n\cos(nx)\,dx & V = u_x(x,t) \end{cases}$$

$$w_n(t) = \frac{2}{\pi}\Big[\sin(nx)\,u_x(x,t)\Big]\Big|_0^\pi - \frac{2n}{\pi}\int_0^\pi u_x(x,t)\cos(nx)\,dx$$

$$\begin{cases} v = \cos nx & dV = u_x(x,t)\,dx \\ du = -n\sin(nx)\,dx & V = u(x,t) \end{cases}$$

$$= \frac{2}{\pi}\sin(nx)\,u_x(x,t)\Big|_0^\pi - \frac{2n}{\pi}\Big[\cos(nx)\,u(x,t)\Big]\Big|_0^\pi - \frac{2n^2}{\pi}\int_0^\pi u(x,t)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\Big[\sin(nx)\,u_x(x,t) - n\cos(nx)\,u(x,t)\Big]\Big|_0^\pi - \frac{2n^2}{\pi}\int_0^\pi u(x,t)\sin(nx)\,dx$$

(***) $\quad w_n(t) = -n^2 u_n(t)$

Substituting from (*) and (***) into (**) yields

$$(\dagger\dagger) \quad u_n''(t) + c^2 n^2 u_n(t) = \begin{cases} 0 & \text{if } n \neq 5, \\ e^t & \text{if } n = 5. \end{cases}$$

From (1a′) and (1′), $y_n(t) = u_n'(t)$ (****). Using the I.C.'s of ($\dagger$) and (*) plus (****) yields

$$(\dagger\dagger\dagger) \begin{cases} u_n(0) = \dfrac{2}{\pi}\displaystyle\int_0^\pi u(x,0)\sin(nx)\,dx = 0 \\ u_n'(0) = \dfrac{2}{\pi}\displaystyle\int_0^\pi u_t(x,0)\sin(nx)\,dx = \dfrac{2}{\pi}\displaystyle\int_0^\pi \sin(3x)\sin(nx)\,dx = \begin{cases} 0 & \text{if } n \neq 3, \\ 1 & \text{if } n = 3. \end{cases} \end{cases}$$

<u>$n \neq 5$ & 3</u>: $\quad u_n''(t) + (cn)^2 u_n(t) = 0, \quad u_n(0) = u_n'(0) = 0 \quad$ implies $u_n(t) \equiv 0$.

<u>$n = 5$</u>: $\quad u_5''(t) + (5c)^2 u_5(t) = e^t, \quad u_5(0) = u_5'(0) = 0$

$$u_5(t) = c_1\cos(5ct) + c_2\sin(5ct) + \frac{1}{1+25c^2}e^t$$

$$u_5'(t) = -5cc_1\sin(5ct) + 5cc_2\cos(5ct) + \frac{e^t}{1+25c^2}$$

$$u_p(t) = Ae^t \Rightarrow Ae^t + (5c)^2Ae^t = e^t \Rightarrow A(1+25c^2) = 1$$

$$\begin{cases} 0 = c_1 + \dfrac{1}{1+25c^2} \\ 0 = 5cc_2 + \dfrac{1}{1+25c^2} \\ \Rightarrow 0 = c_1 - 5cc_2 \\ c_1 = \dfrac{-1}{1+25c^2} \\ \ldots = \dfrac{-1}{} \end{cases}$$

$$u_5(t) = \frac{1}{1+25c^2}\left[e^t - \cos(5ct) - \frac{1}{5c}\sin(5ct)\right]$$

$\underline{n=3}$: $\quad u_3''(t) + (3c)^2 u_3(t) = 0, \quad u_3(0) = 0, \; u_3'(0) = 1$

$$u_3(t) = c_1\cos(3ct) + c_2\sin(3ct) \qquad 0 = c_1$$

$$u_3'(t) = -3cc_1\sin(3ct) + 3cc_2\cos(3ct) \qquad 1 = 3cc_2 \Rightarrow c_2 = \frac{1}{3c}$$

$$u_3(t) = \frac{1}{3c}\sin(3ct)$$

$$\therefore \quad u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin(nx)$$

$$\boxed{u(x,t) = \frac{1}{3c}\sin(3ct)\sin(3x) + \frac{1}{1+25c^2}\left[e^t - \cos(5ct) - \frac{1}{5c}\sin(5ct)\right]\sin(5x)}$$

$\underline{\text{Check}}$: $\quad u_{tt} = -3c\sin(3ct)\sin(3x) + \frac{1}{1+25c^2}\left[e^t + 25c^2\cos(5ct) + 5c\sin(5ct)\right]\sin(5x)$

$$-c^2u_{xx} = 3c\sin(3ct)\sin(3x) + \frac{25c^2}{1+25c^2}\left[e^t - \cos(5ct) - \frac{1}{5c}\sin(5ct)\right]\sin(5x)$$

$$u_{tt} - c^2u_{xx} \overset{\checkmark}{=} e^t\sin(5x)$$

$u(0,t) \overset{\checkmark}{=} 0$ and $u(\pi,t) \overset{\checkmark}{=} 0$ for $t \in (-\infty,\infty)$

$u(x,0) \overset{\checkmark}{=} 0$ and $u_t(x,0) \overset{\checkmark}{=} \sin(3x)$

$$u_t = \cos(3ct)\sin(3x) + \frac{1}{1+25c^2}\left[e^t + 5c\sin(5ct) - \cos(5ct)\right]\sin(5x)$$

Sec. 5.6, pp. 144–145.

#7. Repeat exercise 6 for the damped wave equation

$$u_{tt} - c^2 u_{xx} + r u_t = g(x)\sin(\omega t) \quad \text{for } 0<x<\ell,\ 0<t<\infty,$$

where r is a positive constant. [The boundary and initial conditions satisfied by u are

$$u(0,t) = u(\ell,t) = 0 \quad \text{for } t \geq 0,$$
$$u(x,0) = u_t(x,0) = 0 \quad \text{for } 0 \leq x \leq \ell.$$

For which values of ω can resonance occur?]

We use the method of expansion. The associated homogeneous problem

$$\begin{cases} u_{tt} - c^2 u_{xx} + r u_t = 0 & \text{for } 0<x<\ell,\ 0<t<\infty, \\ u(0,t) = u(\ell,t) = 0 & \text{for } t \geq 0, \end{cases}$$

becomes

$$(\dagger) \quad \begin{cases} T'' + rT' + c^2\lambda T = 0 \\ X'' + \lambda X = 0, \quad X(0) = X(\ell) = 0 \end{cases}$$

when the variables are separated via $u(x,t) = X(x)T(t)$. The eigenvalues and eigenfunctions of the second line of $(\dagger)$ are $\lambda_n = \beta_n^2 = \left(\frac{n\pi}{\ell}\right)^2$ and $X_n(x) = \sin(\beta_n x) = \sin\left(\frac{n\pi x}{\ell}\right)$, $(n = 1, 2, 3, \ldots)$, respectively. By Theorem 3, Sec. 5.4, this set of eigenfunctions is complete. Hence for each fixed $t \in [0,\infty)$:

$$(1) \quad u(x,t) \sim \sum_{n=1}^{\infty} u_n(t)\sin\left(\frac{n\pi x}{\ell}\right) \quad \text{where} \quad u_n(t) = \frac{2}{\ell}\int_0^{\ell} u(x,t)\sin\left(\frac{n\pi x}{\ell}\right)dx, \quad (1')$$

$$(1a) \quad u_t(x,t) \sim \sum_{n=1}^{\infty} y_n(t)\sin\left(\frac{n\pi x}{\ell}\right) \quad \text{where} \quad y_n(t) = \frac{2}{\ell}\int_0^{\ell} u_t(x,t)\sin\left(\frac{n\pi x}{\ell}\right)dx, \quad (1a')$$

Sec. 5.6, pp. 144–145.

#7. (cont.)

(2) $u_{tt}(x,t) \sim \sum_{n=1}^{\infty} v_n(t) \sin\left(\frac{n\pi x}{\ell}\right)$ where $v_n(t) = \frac{2}{\ell}\int_0^{\ell} u_{tt}(x,t) \sin\left(\frac{n\pi x}{\ell}\right) dx$, (2′)

(3) $u_{xx}(x,t) \sim \sum_{n=1}^{\infty} w_n(t) \sin\left(\frac{n\pi x}{\ell}\right)$ where $w_n(t) = \frac{2}{\ell}\int_0^{\ell} u_{xx}(x,t) \sin\left(\frac{n\pi x}{\ell}\right) dx$, (3′)

(4) $g(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{\ell}\right)$ where $B_n = \frac{2}{\ell}\int_0^{\ell} g(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$. (4′)

By computations like those in problem #5 we find the following relationships:

(*) $\quad v_n(t) = u_n''(t)$ and $y_n(t) = u_n'(t)$,

(**) $\quad v_n(t) - c^2 w_n(t) + r y_n(t) = B_n \sin(\omega t)$,

(***) $\quad w_n(t) = -\left(\frac{n\pi}{\ell}\right)^2 u_n(t) = -\lambda_n u_n(t)$.

Substituting from (*) and (***) into (**) produces

(††) $\quad u_n''(t) + c^2\lambda_n u_n(t) + r u_n'(t) = B_n \sin(\omega t)$,

and the boundary/initial conditions of problem #7 imply, via (1′) (1′a) and (*), that

(†††) $\quad u_n(0) = u_n'(0) = 0$.

The general solution of (††) is $u_n(t) = u_n^h(t) + u_n^p(t)$ where u_n^p is the particular solution of (††),

($\square$) $$u_n^p(t) = \frac{B_n\left[(c^2\lambda_n - \omega^2)\sin(\omega t) - r\omega\cos(\omega t)\right]}{(c^2\lambda_n-\omega^2)^2 + r^2\omega^2},$$

and u_n^h is the general solution to the homogeneous differential equation associated with (††):

#7. (cont.)

$$(\square\square)\qquad u_n^h(t) = \begin{cases} e^{-\frac{rt}{2}}\left[a_n\cos(\mu_n t) + b_n \sin(\mu_n t)\right] & \text{if} \quad \mu_n^2 = c^2\lambda_n - \frac{r^2}{4} > 0, \\ e^{-\frac{rt}{2}}\left[a_n + b_n t\right] & \text{if} \quad \mu_n^2 = c^2\lambda_n - \frac{r^2}{4} = 0, \\ e^{-\frac{rt}{2}}\left[a_n\cosh(\mu_n t) + b_n \sinh(\mu_n t)\right] & \text{if} \quad \mu_n^2 = \frac{r^2}{4} - c^2\lambda_n > 0. \end{cases}$$

Applying the I. C.'s $(\dagger\dagger\dagger)$, we determine the arbitrary coefficients a_n and b_n:

$$0 = u_n(0) = u_n^p(0) + u_n^h(0) = \frac{-B_n r\omega}{(c^2\lambda_n - \omega^2)^2 + r^2\omega^2} + a_n$$

$$0 = u_n'(0) = (u_n^p)'(0) + (u_n^h)'(0) = \frac{B_n(c^2\lambda_n - \omega^2)\omega}{(c^2\lambda_n-\omega^2)^2 + r^2\omega^2} + \begin{cases} -\frac{a_n r}{2} + b_n\mu_n & \text{if } \mu_n \neq 0, \\ -\frac{a_n r}{2} + b_n & \text{if } \mu_n = 0. \end{cases}$$

$$(\square\square\square)\quad \begin{cases} \text{Thus} \quad a_n = \dfrac{B_n r\omega}{(c^2\lambda_n - \omega^2)^2 + r^2\omega^2} \\[2ex] \text{and} \quad b_n = \dfrac{B_n(\omega^2 - c^2\lambda_n + r^2/2)\omega}{\mu_n\left[(c^2\lambda_n-\omega^2)^2 + r^2\omega^2\right]} \qquad (\text{if } \mu_n \neq 0) \end{cases}$$

If $u_n(t) = u_n^h(t) + u_n^p(t)$ with $(\square)$, $(\square\square)$, and $(\square\square\square)$ then

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\left(\frac{n\pi x}{\ell}\right)$$

solves the problem of exercise #7. Because $r > 0$ and $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$, $(n = 1, 2, 3, \ldots)$, grows sufficiently rapidly, $\sum_{n=1}^{\infty} |u_n(t)| \leq B < \infty$ for all $t \geq 0$.

Hence resonance cannot occur.

Sec. 5.6, pp. 144–145.

#9. Use the method of subtraction to solve

$$\begin{cases} u_{tt} - 9u_{xx} = 0 & \text{for } 0 \le x \le 1,\ 0 \le t < \infty, \\ u(0,t) = h,\ u(1,t) = k & \text{for } t \ge 0, \\ u(x,0) = u_t(x,0) = 0 & \text{for } 0 \le x \le 1, \end{cases}$$

where h and k are given constants.

Let $v(x,t) = u(x,t) - f(x)$ where $u = u(x,t)$ is a solution to the problem above and f is a linear function satisfying $f(0) = h$ and $f(1) = k$. [Explicitly $f(x) = (k-h)x + h$.] Then v solves

$$\begin{cases} v_{tt} - 9v_{xx} = 0 & \text{for } 0 \le x \le 1,\ 0 \le t < \infty, \\ v(0,t) = v(1,t) = 0 & \text{for } t \ge 0, \\ v(x,0) = -f(x) \text{ and } v_t(x,0) = 0 & \text{for } 0 \le x \le 1. \end{cases}$$

Standard separation of variables techniques yield a (formal) solution

$$v(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)\cos(3n\pi t)$$

to the PDE and the 3 homogeneous initial/boundary conditions. If the c_n's are chosen to be the Fourier sine coefficients of $-f$ then we satisfy the nonhomogeneous initial condition as well.

$$c_n = 2\int_0^1 -f(x)\sin(n\pi x)\,dx = 2(h-k)\int_0^1 x\sin(n\pi x)\,dx - 2h\int_0^1 \sin(n\pi x)\,dx$$

$$c_n = 2(h-k)\left[\frac{-x\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{(n\pi)^2}\right]\Bigg|_0^1 + \frac{2h\cos(n\pi x)}{n\pi}\Bigg|_0^1 = \frac{2\left[(-1)^n k - h\right]}{n\pi}$$

By Theorem 4, Sec. 5.4, $-\left[(k-h)x + h\right] = \displaystyle\sum_{n=1}^{\infty} \frac{2\left[(-1)^n k - h\right]}{n\pi}\sin(n\pi x)$ for $0 < x < 1$.

Sec. 5.6, pp. 144-145.

#9. (cont.) Therefore

$$u(x,t) = f(x) + v(x,t) = (k-h)x + h + \sum_{n=1}^{\infty} \frac{2[(-1)^n k - h]}{n\pi} \sin(n\pi x)\cos(3n\pi t)$$

is the solution to the problem in exercise #9.