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#1. Solve $u_{xx} + u_{yy} = 0$ in the rectangle $0<x<a$, $0<y<b$ with the following (Neumann) boundary conditions:

$$u_x(0,y) = -a \quad \text{and} \quad u_x(a,y) = 0 \quad \text{for} \quad 0 \le y \le b;$$
$$u_y(x,0) = b \quad \text{and} \quad u_y(x,b) = 0 \quad \text{for} \quad 0 \le x \le a.$$

Preliminary notes: Note that the nonhomogeneous boundary conditions on the edges $x=0$ and $y=0$ prevent a direct separation of variables technique from succeeding. Therefore we use the method of shifting the (nonhomogeneous) boundary data to the PDE by subtracting a known function that satisfies them. It is readily apparent that a quadratic polynomial in the variables x and y can be found that satisfies the boundary data.

Let $v(x,y) = u(x,y) - (Ax^2 + Bxy + Cy^2 + Dx + Ey + F)$ where the constants $A, B, \ldots, F$ will be chosen later. Then in the rectangle $0<x<a$, $0<y<b$,

$$(*) \qquad v_{xx} + v_{yy} = \overbrace{u_{xx} + u_{yy}}^{0} - 2A - 2C = -2(A+C).$$

On the boundaries we want to choose $A, B, \ldots, F$ so as to obtain homogeneous conditions. Therefore

$$0 = v_x(0,y) = u_x(0,y) - (2Ax + By + D)\Big|_{x=0} = -a - By - D \quad \text{for } 0 \le y \le b$$

implies $\boxed{B=0 \text{ and } D=-a}$;

$$0 = v_x(a,y) = u_x(a,y) - (2Ax + By + D)\Big|_{x=a} = 0 - 2Aa - \cancel{By}^{\,0} - \cancel{D}^{\,-a} \quad \text{for } 0 \le y \le b$$

implies $\boxed{A = \frac{1}{2}}$;

#1 (cont.)

$$0 = v_y(x,0) = u_y(x,0) - (Bx + 2Cy + E)\Big|_{y=0} = b - \cancel{B}^{0}x - E \quad \text{for } 0 \le x \le a$$

implies $\boxed{E = b}$;

$$0 = v_y(x,b) = u_y(x,b) - (Bx + 2Cy + E)\Big|_{y=b} = 0 - \cancel{B}^{0}x - 2Cb - \cancel{E}^{b}$$

implies $\boxed{C = -\frac{1}{2}}$.

Therefore if we set $v(x,y) = u(x,y) - \left(\frac{1}{2}x^2 - \frac{1}{2}y^2 + ax + by + F\right)$ then by $(*)$ and the B.C. calculations above, the original problem is transformed into

$$(**)\begin{cases} v_{xx} + v_{yy} = 0 & \text{for } 0<x<a,\ 0<y<b, \\ v_x(0,y) = v_x(a,y) = v_y(x,0) = v_y(x,b) = 0 & \text{for } 0 \le x \le a,\ 0 \le y \le b \end{cases}$$

Clearly any constant function solves $(**)$. By exercise #12(b), Sec. 6.4 this is the only (type of) solution to $(**)$. Therefore

$$\text{constant} = v(x,y) = u(x,y) - \left(\tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 + ax + by + F\right)$$

so $$\boxed{u(x,y) = \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 + ax + by + F'}$$

where F' is any constant.

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#2 Prove that the eigenfunctions $\{\psi_{m,n}(y,z)\}_{\substack{m=1\\n=1}}^{\infty} = \{\sin(my)\sin(nz)\}_{\substack{m=1\\n=1}}^{\infty}$ are orthogonal on the square $0<y<\pi$, $0<z<\pi$.

Let m and n, m' and n', be positive integers. We compute the inner product of $\psi_{m,n}$ and $\psi_{m',n'}$ on the square:

$$(\psi_{m,n}, \psi_{m',n'}) = \int_0^{\pi}\int_0^{\pi} \psi_{m,n}(y,z)\,\psi_{m',n'}(y,z)\,dy\,dz$$

$$= \int_0^{\pi}\int_0^{\pi} \sin(my)\sin(nz)\sin(m'y)\sin(n'z)\,dy\,dz$$

$$= \int_0^{\pi} \sin(nz)\sin(n'z)\left(\int_0^{\pi}\sin(my)\sin(m'y)\,dy\right)dz$$

$$= \underbrace{\left(\int_0^{\pi}\sin(nz)\sin(n'z)\,dz\right)}_{0 \text{ if } n\neq n'}\underbrace{\left(\int_0^{\pi}\sin(my)\sin(m'y)\,dy\right)}_{0 \text{ if } m\neq m'} \quad \text{(see (2), p. 101)}$$

Therefore $(\psi_{m,n}, \psi_{m',n'}) = 0$ if $(m,n) \neq (m',n')$.

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#3 Find the harmonic function $u = u(x,y)$ in the square $D = \{(x,y) : 0 < x < \pi,\ 0 < y < \pi\}$ satisfying the boundary conditions

$$u_y(x,0) = u_y(x,\pi) = 0 \quad \text{for} \quad 0 \le x \le \pi,$$

$$u(0,y) = 0 \quad \text{and} \quad u(\pi,y) = \cos^2(y) = \frac{1+\cos(2y)}{2} \quad \text{for} \quad 0 \le y \le \pi.$$

Separation of variables for the Laplace equation $\nabla^2 u = 0$ in D, and applying the homogeneous B.C.'s gives:

$$\begin{cases} X'' - \lambda X = 0, & X(0) = 0, \\ Y'' + \lambda Y = 0, & Y'(0) = Y'(\pi) = 0. \end{cases}$$

Hence $\lambda = \lambda_n = n^2$, $Y_n(y) = \cos(ny)$, and $X_n(x) = \begin{cases} A_n \sinh(nx), & n \ne 0 \\ A_0 x & \text{if } n = 0, \end{cases}$

(Here $n = 0, 1, 2, \ldots$). The formal series solution is

$$u(x,y) = A_0 x + \sum_{n=1}^{\infty} A_n \sinh(nx)\cos(ny).$$

We must choose the undetermined coefficients $A_0, A_1, A_2, \ldots$ so as to satisfy the nonhomogeneous boundary condition:

$$\frac{1}{2} + \frac{1}{2}\cos(2y) = u(\pi,y) = \pi A_0 + \sum_{n=1}^{\infty} A_n \sinh(n\pi)\cos(ny) \quad \text{for} \quad 0 \le y \le \pi.$$

Comparing coefficients on the left and right-most members of the above identity we see that $\frac{1}{2} = \pi A_0$, $\frac{1}{2} = A_2 \sinh(2\pi)$, and $A_n = 0$ for $n = 1, 3, 4, 5, \ldots$

Thus

$$u(x,y) = A_0 x + A_2 \sinh(2x)\cos(2y) = \boxed{\frac{x}{2\pi} + \frac{\sinh(2x)\cos(2y)}{2\sinh(2\pi)}}.$$

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Example 1 (p. 155) Solve

(1) $$u_{xx} + u_{yy} = 0$$

in the rectangle $D = \{(x,y) : 0<x<a,\ 0<y<b\}$ with the boundary conditions

$$(*) \quad \begin{cases} u(x,b) = g(x) \quad \text{and} \quad u_y(x,0) + u(x,0) = h(x) \quad \text{for} \quad 0 \le x \le a; \\ u(0,y) = j(y) \quad \text{and} \quad u_x(a,y) = k(y) \quad \text{for} \quad 0 \le y \le b. \end{cases}$$

#5. Solve example 1 in the case $g = h = k = 0$ but $j = j(y)$ is an "arbitrary" function.

We separate variables. Let $u(x,y) = X(x)Y(y)$. Then (1) implies

$$\frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} = \text{constant} = \lambda,$$

and hence, using the B.C.'s (*), we have

(2) $$X''(x) - \lambda X(x) = 0 \quad , \quad X'(a) = 0$$

(3) $$Y''(y) + \lambda Y(y) = 0 \, , \quad Y(b) = Y'(0) + Y(0) = 0 \, .$$

Case 1: The eigenvalues λ of (3) are positive, say $\lambda = \beta^2 > 0$.
The general solution of the ODE in (3) is then $Y(y) = c_1 \cos(\beta y) + c_2 \sin(\beta y)$, and the B.C.'s imply

$$0 = Y'(0) + Y(0) = \beta c_2 + c_1$$
$$0 = Y(b) = c_1 \cos(\beta b) + c_2 \sin(\beta b) .$$

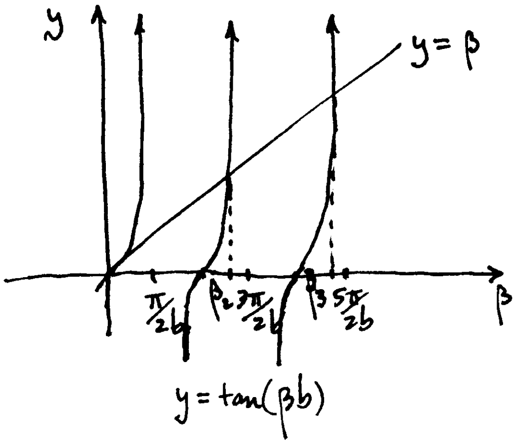
In order for a nontrivial solution to the homogeneous linear system of equations in the variables c_1 and c_2 to exist, the coefficient matrix must have a zero determinant, i.e.

$$0 = \begin{vmatrix} 1 & \beta \\ \cos(\beta b) & \sin(\beta b) \end{vmatrix} = \sin(\beta b) - \beta \cos(\beta b).$$

#5 (cont.) If $\cos(\beta b) = 0$ then $\pm 1 = \sin(\beta b) = 0$, a contradiction. Thus $\cos(\beta b) \neq 0$ and hence

$$\beta = \tan(\beta b)$$

is the condition that positive eigenvalues $\lambda = \beta^2$ must satisfy.



Clearly there is an infinite sequence $\{\beta_n\}$ of intersection points satisfying

$$\beta_n \in \left(\frac{(n-1)\pi}{b}, \frac{(n-\frac{1}{2})\pi}{b} \right) \quad (n = 2, 3, \ldots)$$

with $\lim_{n\to\infty} \left[\beta_n - \frac{(n-\frac{1}{2})\pi}{b} \right] = 0$.

There is an intersection point $\beta_1 \in (0, \frac{\pi}{2b})$ if and only if

$$b = (\text{slope of } y = \tan(\beta b) \text{ at } \beta = 0) < (\text{slope of } y = \beta \text{ at } \beta = 0) = 1.$$

Case 2: $\lambda = 0$ is an eigenvalue of (3).

In this case the general solution of the ODE in (3) is $Y(y) = c_1 y + c_2$ and then B.C.'s imply

$$0 = Y'(0) + Y(0) = c_1 + c_2$$
$$0 = Y(b) = c_1 b + c_2 .$$

Then $0 = $ determinant of the coefficient matrix $= \begin{vmatrix} 1 & 1 \\ b & 1 \end{vmatrix} = 1 - b$ is the condition that must be satisfied in order for $\lambda = 0$ to be an eigenvalue of (3).

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#5 (cont.) Case 3: $\lambda = -\beta^2 < 0$ is an eigenvalue of (3).

In this case $Y(y) = c_1 \cosh(\beta y) + c_2 \sinh(\beta y)$ is the general solution to the ODE in (3). The B.C.'s yield

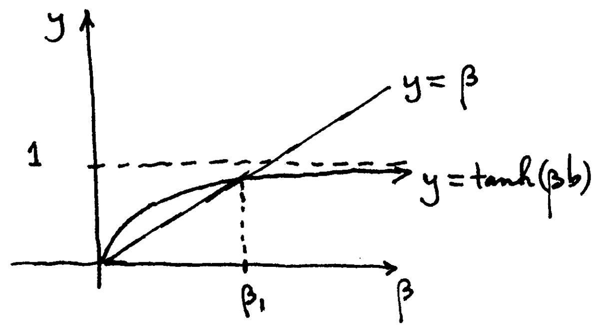
$$0 = Y'(0) + Y(0) = \beta c_2 + c_1$$
$$0 = Y(b) = c_1 \cosh(\beta b) + c_2 \sinh(\beta b).$$

In order for a nontrivial solution to this system to exist, we must have

$$0 = \begin{vmatrix} 1 & \beta \\ \cosh(\beta b) & \sinh(\beta b) \end{vmatrix} = \sinh(\beta b) - \beta \cosh(\beta b).$$

I.e.

$$\beta = \tanh(\beta b).$$



There is an intersection point $\beta_1 > 0$ if and only if

$$b = (\text{slope of } y = \tanh(\beta b) \text{ at } \beta = 0) > (\text{slope of } y = \beta \text{ at } \beta = 0) = 1.$$

Summarizing: There is an infinite sequence of eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ of (3) with $\lambda_n \to +\infty$ such that if:

$\underline{b < 1}$ all the eigenvalues are positive, $\lambda_n = \beta_n^2 > 0$ $(n = 1, 2, 3, \ldots)$, and the eigenfunctions are (up to a constant multiple)

$$(**) \qquad Y_n(y) = \sin(\beta_n y) - \beta_n \cos(\beta_n y) \; ;$$

#5 (cont.) <u>$b=1$</u> zero is an eigenvalue, $\lambda_1 = 0$, and the remaining eigenvalues are positive, $\lambda_n = \beta_n^2 > 0$ $(n = 2, 3, \dots)$. The corresponding eigenfunctions are (up to a constant multiple) $Y_1(y) = y - 1$ and Y_n $(n = 2, 3, \dots)$ given by (**);

<u>$b>1$</u> one eigenvalue is negative, $\lambda_1 = -\beta_1^2 < 0$, and the remaining eigenvalues are positive, $\lambda_n = \beta_n^2 > 0$ $(n = 2, 3, \dots)$. The corresponding eigenfunctions are $Y_1(y) = \sinh(\beta_1 y) - \beta_1 \cosh(\beta_1 y)$ and Y_n $(n = 2, 3, \dots)$ given again by (**);

Returning to (2), we see that the corresponding solutions are (up to a constant multiple)

$$X_n(x) = \cosh(\beta_n(x-a)) \quad \text{if} \quad \lambda_n = \beta_n^2 > 0 \ (\text{Case 1 and } b<1),$$

$$X_1(x) = 1 \quad \text{if} \quad \lambda_1 = 0 \quad (\text{Case 2 and } b=1),$$

$$X_1(x) = \cos(\beta_1(x-a)) \quad \text{if} \quad \lambda_1 = -\beta_1^2 < 0 \quad (\text{Case 3 and } b>1).$$

Therefore $u(x,y) = \sum_{n=1}^{\infty} c_n X_n(x) Y_n(y)$ is a (formal) solution to the PDE (1) and the homogeneous B.C.'s in (*) for any choice of constants $c_1, c_2, c_3, \dots$ To satisfy the nonhomogeneous B.C.

$$j(y) = u(0,y) = \sum_{n=1}^{\infty} c_n X_n(0) Y_n(y) \qquad \text{for } 0 \le y \le b,$$

we choose the constants so that $c_n X_n(0)$ is the n^{th} Fourier coefficient of j with respect to the orthogonal set $\{Y_n\}_{n=1}^{\infty}$ on $(0,b)$:

$$c_n X_n(0) = \frac{(j, Y_n)}{(Y_n, Y_n)} = \frac{\int_0^b j(y) Y_n(y)\, dy}{\int_0^b Y_n^2(y)\, dy} \qquad (n = 1, 2, 3, \dots).$$

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#7. (a) Find the harmonic function in the semi-infinite strip $\{(x,y): 0 \le x \le \pi, 0 \le y < \infty\}$ that satisfies the boundary conditions

(1) $\quad 0 = u(0,y) = u(\pi,y) \quad$ for $\quad 0 \le y < \infty$,

(2) $\quad h(x) = u(x,0) \quad$ for $\quad 0 \le x \le \pi$,

(3) $\quad \lim_{y\to\infty} u(x,y) = 0 \quad$ for each $x \in [0,\pi]$.

(b) What would go awry if we omitted the condition at infinity?

(a) We seek a solution to

(4) $\quad u_{xx} + u_{yy} = 0 \quad$ for $0<x<\pi$, $0<y<\infty$,

continuous on the semi-infinite strip, and satisfying (1), (2), and (3). We separate variables; let $u(x,y) = X(x)Y(y)$. Substituting in (4) gives

$$-\frac{X''}{X} = \frac{Y''}{Y} = \text{constant} = \lambda,$$

and applying (1) and (3) gives

(5) $\quad X'' + \lambda X = 0, \qquad X(0) = X(\pi) = 0,$

(6) $\quad Y'' - \lambda Y = 0, \qquad \lim_{y\to\infty} Y(y) = 0.$

The eigenvalues of (5) are $\lambda_n = n^2$ $(n=1,2,3,\dots)$ and the corresponding eigenfunctions are $X_n(x) = \sin(nx)$ $(n=1,2,3,\dots)$. The solutions of (6) are then (up to a constant multiple) $Y_n(y) = e^{-ny}$ $(n=1,2,3,\dots)$. Thus

(7) $$u(x,y) = \sum_{n=1}^{\infty} b_n Y_n(y) X_n(x) = \sum_{n=1}^{\infty} b_n e^{-ny} \sin(nx)$$

is a formal solution to (1)–(3)–(4). We seek constants $b_1, b_2, \dots$

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#7 (cont.) such that (2) is satisfied, i.e.

$$h(x) = u(x,0) = \sum_{n=1}^{\infty} b_n Y_n(0) X_n(x) = \sum_{n=1}^{\infty} b_n \sin(nx) , \qquad 0 \le x \le \pi.$$

Choose the coefficients b_n $(n=1,2,3,\dots)$ to be the n^{th} Fourier coefficient of h with respect to the orthogonal set $\{X_n\}_{n=1}^{\infty}$ on $(0,\pi)$:

$$(8) \qquad b_n = \frac{(h, \sin(nx))}{(\sin(x), \sin(x))} = \frac{2}{\pi}\int_0^{\pi} h(x)\sin(nx)\,dx , \quad (n=1,2,3,\dots).$$

Therefore (7)–(8) determines the solution to (1)–(2)–(3)–(4).

(b) If (3) is omitted then we would obtain (via the methods above) a formal solution

$$u(x,y) = \sum_{n=1}^{\infty} (a_n e^{ny} + b_n e^{-ny}) \sin(nx)$$

to (1)–(2)–(4) where

$$a_n + b_n = \frac{(h, \sin(nx))}{(\sin(nx), \sin(nx))} = \frac{2}{\pi}\int_0^{\pi} h(x)\sin(nx)\,dx, \quad (n=1,2,3,\dots).$$

Thus, on the surface, we would appear to lose uniqueness of the solution.

Additional Problem: Solve $\nabla^2 u = 0$ in the cube $0<x<\pi, 0<y<\pi, 0<z<\pi$, given that $u(x,y,0) = \sin(x)\sin^3(y)$ for $0 \le x \le \pi, 0 \le y \le \pi$, and that u satisfies homogeneous Dirichlet boundary conditions on the other five faces of the cube.

We seek a solution to $u_{xx} + u_{yy} + u_{zz} = 0$ in the cube of the form $u(x,y,z) = X(x)Y(y)Z(z)$. Substituting in the PDE yields first

$$-\frac{X''}{X} = \frac{Y''}{Y} + \frac{Z''}{Z} = \text{constant} = \lambda_1$$

and then

$$-\frac{Y''}{Y} = \frac{Z''}{Z} - \lambda_1 = \text{constant} = \lambda_2 .$$

This leads to the coupled system of ODE's:

$$(*) \quad \begin{cases} X'' + \lambda_1 X = 0, & 0<x<\pi, \\ Y'' + \lambda_2 Y = 0, & 0<y<\pi, \\ Z'' - (\lambda_1+\lambda_2)Z = 0, & 0<z<\pi. \end{cases}$$

The homogeneous Dirichlet B.C.'s then imply

$$(**) \quad \begin{cases} X(0) = X(\pi) = 0, \\ Y(0) = Y(\pi) = 0, \\ Z(\pi) = 0. \end{cases}$$

The X and Y eigenvalue problems in $(*)$-$(**)$ have solutions:

$$(***) \quad \begin{cases} \lambda_1 = n^2 \text{ and } X_n(x) = \sin(nx) & (n = 1,2,3,\dots) \\ \lambda_2 = m^2 \text{ and } Y_m(y) = \sin(my) & (m = 1,2,3,\dots). \end{cases}$$

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Additional Problem (cont.) From (*), (**), and (***) it follows that (up to a constant multiple) $Z_{m,n}(z) = \sinh\left(\sqrt{m^2+n^2}\,(z-\pi)\right)$. Therefore

$$u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{m,n} \sinh\left(\sqrt{m^2+n^2}\,(z-\pi)\right)\sin(nx)\sin(my)$$

is a formal solution to $\nabla^2 u = 0$ in the cube which satisfies the homogeneous Dirichlet B.C.'s on the five faces other than $z=0$. We want to determine the constants $c_{m,n}$ $(m,n=1,2,3,\dots)$ so that the nonhomogeneous (Dirichlet) B.C. is satisfied. Making use of the identity $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta)$, we thus require, for $0 \le x,y \le \pi$,

$$\frac{3}{4}\sin(x)\sin(y) - \frac{1}{4}\sin(x)\sin(3y) = u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} -c_{m,n}\sinh\left(\pi\sqrt{m^2+n^2}\right)\sin(nx)\sin(m$$

By orthogonality of $\{\sin(nx)\}_{n=1}^{\infty}$ on $(0,\pi)$, it follows that

$$\frac{3}{4} = -c_{1,1}\sinh(\pi\sqrt{2})$$

$$-\frac{1}{4} = -c_{3,1}\sinh(\pi\sqrt{10})$$

and $c_{m,n} = 0$ otherwise. Consequently, the solution is

$$\boxed{u(x,y,z) = \frac{\sinh\left(\sqrt{10}\,(z-\pi)\right)\sin(x)\sin(3y)}{4\sinh(\pi\sqrt{10})} - \frac{3\sinh\left(\sqrt{2}\,(z-\pi)\right)\sin(x)\sin(y)}{4\sinh(\pi\sqrt{2})}}.$$