NAME KEY

Math 1212
Test 4
Fall 2015

You have 50 minutes to complete this test. You must *show all work* to receive full credit. Work any 6 of the following 7 problems. Clearly **CROSS OUT** the problem you do not wish me to grade. Each problem is worth 16 points, and you get 4 points for free, for a total of 100 points. The answers will be posted on the electronic reserves tomorrow afternoon.

1. Find the area of the region bounded by the curves $y = x^3 + 3x^2$ and $y = 4x$. Be sure to sketch a graph first!

Intersection points where

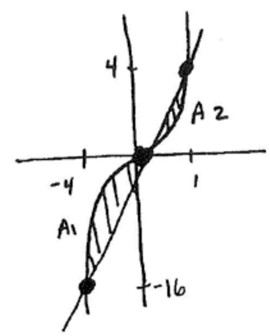
$x^3 + 3x^2 = 4x$

$x^3 + 3x^2 - 4x = 0$

$x(x^2 + 3x - 4) = 0$

$x(x+4)(x-1) = 0$

$\left.\begin{array}{ll} x=0 & (0,0) \\ x=-4 & (-4,-16) \\ x=1 & (1,4) \end{array}\right\}$ use $y=4x$



$\text{Area} = A_1 + A_2$

$= \int_{-4}^{0} (x^3 + 3x^2 - 4x)\,dx + \int_0^1 (4x - x^3 - 3x^2)\,dx$

$= \left(\frac{1}{4}x^4 + x^3 - 2x^2\right)\Big|_{-4}^{0} + \left(2x^2 - \frac{1}{4}x^4 - x^3\right)\Big|_0^1 = [0 - (64 - 64 - 32)] + [2 - \frac{1}{4} - 1]$

$= 32 + \frac{3}{4} = \frac{131}{4}$

2. Compute both first-order partial derivatives of $f(x, y) = \dfrac{xy^2}{x^2y^3 + 1}$.

$$f_x = \frac{(y^2)(x^2y^3+1) - (xy^2)(2xy^3)}{(x^2y^3+1)^2}$$

$$f_y = \frac{(2xy)(x^2y^3+1) - (xy^2)(3x^2y^2)}{(x^2y^3+1)^2}$$

3. Find and classify the critical points of $f(x, y) = 4xy - 2x^4 - y^2 + 4x - 2y$.

$$\left.\begin{array}{l} f_x = 4y - 8x^3 + 4 = 0 \to y - 2x^3 + 1 = 0 \to y = 2x^3 - 1 \\ f_y = 4x - 2y - 2 = 0 \to 2x - y - 1 = 0 \to y = 2x - 1 \end{array}\right\} 2x^3 - 1 = 2x - 1$$

$2x^3 - 2x = 0$

$x^3 - x = 0$

$x(x^2 - 1) = 0$

$x = 0 \quad (0, -1)$ use $y = 2x - 1$ to get y.

$x = 1 \quad (1, 1)$

$x = -1 \quad (-1, -3)$

$$\left.\begin{array}{l} f_{xx} = -24x^2 \\ f_{yy} = -2 \\ f_{xy} = 4 \end{array}\right\} \begin{array}{l} D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 \\ \quad = 48x^2 - 16 \end{array}$$

$D(0,-1) = -16 < 0$, so $(0,-1)$ is a saddle point

$D(1,1) = 48 - 16 > 0$, $f_{xx}(1,1) = -24 < 0$, so $(1,1)$ is a maximum

$D(-1,-3) = 48 - 16 > 0$, $f_{xx}(-1,-3) = -24 < 0$, so $(-1,-3)$ is a maximum

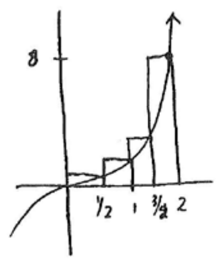
4. Suppose p_1 and p_2 are the prices of two products. Also suppose $D_1(p_1, p_2) = 2000 + \dfrac{100}{p_1 + 2} - 25p_2$ and $D_2(p_1, p_2) = 1500 + \dfrac{p_2}{p_1 + 7}$ are the demand functions for the two products (quantities). Are the two products competitive (substitutes) or complementary? Give an example of two products which might behave in this way.

$$\frac{\partial D_1}{\partial p_2} = -25 < 0 \qquad \frac{\partial D_2}{\partial p_1} = 0 + \frac{(0)(p_1+7) - p_2(1)}{(p_1+7)^2} = \frac{-p_2}{(p_1+7)^2} < 0$$

Both are negative, so the products are complementary.

Two products that might behave this way are cameras & film...

5. Using four rectangles, approximate the area between $f(x) = x^3$ and the x-axis for $0 \le x \le 2$. Be sure to draw a picture!



$$A \approx R_1 + R_2 + R_3 + R_4$$
$$\approx \frac{1}{2}\left(\frac{1}{2}\right)^3 + \frac{1}{2}(1)^3 + \frac{1}{2}\left(\frac{3}{2}\right)^3 + \frac{1}{2}(2)^3$$
$$\approx \frac{1}{16} + \frac{8}{16} + \frac{27}{16} + \frac{64}{16}$$
$$\approx \frac{100}{16}$$
$$\approx \frac{25}{4} \text{ square units}$$

6. Suppose the time X a student must spend waiting in line at the Cashier's Office is a random variable that is exponentially distributed with density function

$$f(x) = \begin{cases} \frac{1}{4}e^{\frac{-x}{4}} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$ where x is the number of minutes a randomly selected student is in line. Find the probability that a student will have to stand in line at least 8 minutes.

$$P(X \geq 8) = \int_8^\infty f(x)\,dx = \int_8^\infty \frac{1}{4}e^{-\frac{1}{4}x}\,dx$$
$$= \lim_{n\to\infty} \int_8^n \frac{1}{4}e^{-\frac{1}{4}x}\,dx$$
$$= \lim_{n\to\infty} \left[-e^{-\frac{1}{4}x}\Big|_8^n\right]$$
$$= \lim_{n\to\infty} \left[-e^{-n/4} + e^{-2}\right]$$

as $n \to \infty$, this term $\to 0$

$$= e^{-2}$$

7. A manager has $60,000 to spend on development and promotion of a new product. If x thousand dollars are spent on development and y thousand dollars are spent on promotion, $f(x,y) = 20x^{\frac{3}{2}}y$ units of the product will be sold. How should the budget be allocated in order to maximize sales?

$x + y = 60$ (constraint)

$F(x,y) = 20x^{3/2}y - \lambda(x+y-60)$

$F_x = 30x^{1/2}y - \lambda = 0$

$F_y = 20x^{3/2} - \lambda = 0$

$F_\lambda = -x - y + 60 = 0$

$30x^{1/2}y = 20x^{3/2}$

$y = \frac{2}{3}\frac{x^{3/2}}{x^{1/2}} = \frac{2}{3}x$

$-x - (\frac{2}{3}x) + 60 = 0$

$-\frac{5}{3}x + 60 = 0$

$\frac{5}{3}x = 60$

$x = 36$

$y = \frac{2}{3}(36) = 24$

To maximize sales, allocate $36.000 to development and $24,000 to promotion.

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