You have 50 minutes to complete this test. You must show all work to receive full credit. Work any 7 of the following 8 problems. Clearly cross out the problem you do not wish me to grade. Each problem is worth 14 points, and you get 2 points for free, for a total of 100 points. If you have any questions, please come to the front and ask.

1. Using the definition of the derivative, find \( f'(x) \) if \( f(x) = x^2 - 3x + 1 \).

\[
\begin{align*}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - 3(x+h) + 1 - (x^2 - 3x + 1)}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h + 1 - x^2 + 3x - 1}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h \to 0} (2x+1) = 2x - 3
\end{align*}
\]

2. Evaluate the following limits. If any of them do not exist, explain why not ("because it's undefined" or "denominator is zero" are not sufficient explanations).

(a) \( \lim_{x \to 3} \frac{x+1}{x+5} = \frac{5+1}{5+5} = \frac{6}{10} = \frac{3}{5} \)

(b) \( \lim_{x \to 3} \frac{9-x^2}{x-3} = \lim_{x \to 3} \frac{-x^2+9}{x-3} = \lim_{x \to 3} \frac{-x+3}{x-3} = -6 \)

(c) \( \lim_{x \to 2} \frac{x+1}{x+2} \) filling in gives \( \frac{x+1}{x+2} \), must use a chart

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.1</td>
<td>1.1</td>
</tr>
<tr>
<td>-2.01</td>
<td>1.01</td>
</tr>
<tr>
<td>-2.001</td>
<td>1.001</td>
</tr>
</tbody>
</table>

So \( \lim_{x \to -2} \frac{x+1}{x+2} = \infty \)
3. Suppose all $x$ units of a product can be sold if the price is set at $p(x) = -x^2 + 4x + 10$. Also assume that the total cost to produce all $x$ units is $C(x) = \frac{1}{3} x^3 + 2x + 39$.

(a) Find an equation for profit when $x$ units are produced.
(b) Using marginal analysis, estimate the change in profit derived from the production and sale of the $5^{th}$ unit.

\[
\begin{align*}
\text{Profit} &= \text{Revenue} - \text{Cost} \\
&= \text{price} \cdot \text{quantity} - \text{Cost} \\
\mathcal{P}(x) &= (-x^2 + 4x + 10)(x) - (\frac{1}{3} x^3 + 2x + 39) \\
\mathcal{P}(x) &= -x^3 + \frac{4}{3} x^2 + 8x - 39
\end{align*}
\]

(b) Estimate, so use derivative.

\[
\begin{align*}
\text{Change in profit} &\approx \mathcal{P}'(x) = -3x^2 + \frac{22}{3} x + 8 \\
\text{Change for 5th unit} &\approx \mathcal{P}'(4) = -3(16) + \frac{88}{3} + 8 \\
&= \frac{88}{3} - 40 = -\frac{2}{3} \\
\text{(Profit decreases about} &\quad \text{10.67)}
\end{align*}
\]

4. Find $f'(x)$ if:

\[a) \quad f(x) = \frac{2x-3}{x^3} \]

\[
\begin{align*}
f'(x) &= \frac{(2)(x^3) - (2x-3)(3x^2)}{x^6} \\
&= \frac{-4x^3 + 9x^2}{x^6} \\
&= -\frac{4x + 9}{x^4}
\end{align*}
\]

\[b) \quad f(x) = x^3 - \frac{1}{3} x^5 + 2x^{\frac{3}{2}} + \sqrt{x} \]

\[
\begin{align*}
f'(x) &= 3x^2 - \frac{4}{5} x^{-\frac{2}{5}} + \frac{3}{2} x^{-\frac{1}{2}}
\end{align*}
\]
5. Find the equation of the line tangent to \( f(x) = (2x+1)(x^2 - x + 3) \) at the point where \( x = 0 \).

\[
\text{Slope: } \quad f'(x) = (2)(x^2 - x + 3) + (2x+1)(2x-1) \\
\quad m = f'(0) = (2)(3) + (1)(-1) = 6 - 1 = 5
\]

\[
\text{Point: } \quad x=0 \\
\quad y = f(0) = (1)(3) = 3 \\
\quad (0,3)
\]

\[
\text{Line: } \quad y - 3 = 5(x-0) \\
\quad y = 5x + 3
\]

6. Graph the function \( f(x) = \begin{cases} 
  x^2 - 3x + 2 & \text{if } x \leq 3 \\
  x + 1 & \text{if } x > 3
\end{cases} \). Your graph should be clearly labeled and large enough for me to see everything easily.

(a) For what values of \( x \) is \( f(x) \) discontinuous? \( x = 3 \)

(b) Find \( \lim_{x \to 0} f(x) \).

(c) Find \( \lim_{x \to 3^-} f(x) \).

(d) Find \( \lim_{x \to 3^+} f(x) \).

(e) Find \( \lim_{x \to 3} f(x) \). DNE

\[
\text{Note: to get the point} \\
\text{where } x = 3, \text{ fill in to top:} \\
f(3) = 3^2 - 3 \cdot 3 + 2 = 2 \\
\text{to get the hole, fill } x = 3 \text{ into the other part:} \\
y = 3 + 1 = 4 \\
\text{hole at } (3,4)
\]
7. Suppose that the total cost to produce \( x \) units of a commodity is given by
\[
C(x) = 2x^2 - 12x + 30 \text{ dollars.}
\]
Using calculus, determine how many units should be produced in order to minimize cost. What is the minimum cost?

Notice the graph of \( C(x) \) is a parabola that opens up. To find the minimum, we can find the value of \( x \) that makes \( C'(x) = 0 \), since that means the tangent line has slope zero (which happens at the bottom).

\[
C'(x) = 4x - 12 = 0,
\]
so \( x = 3 \) will give the minimum cost.

This minimum cost is then
\[
C(3) = 2(9) - 36 + 30 = 6
\]

8. Find the derivative of
\[
y = \frac{(3x+1)^{3}}{\sqrt{x^2 + 10x^4}(2x^4 - 6)}
\]
\[
y' = \frac{9x^2 + 6x + 1}{(x^{2/3} + 10x^3)(2x^4 - 6)}
\]

\[
y' = (18x + 6)\left[ (x^{2/3} + 10x^3)(2x^4 - 6) \right]'
\]
\[
- (9x^2 + 6x + 1)\left[ (x^{2/3} + 10x^3)(2x^4 - 6) \right]'
\]
\[
\frac{\left[ (x^{2/3} + 10x^3)(2x^4 - 6) \right]^2}{\left[ (x^{2/3} + 10x^3)(2x^4 - 6) \right]^2}
\]

\[
y' = (18x + 6)\left[ (x^{2/3} + 10x^3)(2x^4 - 6) \right]
\]
\[
- (9x^2 + 6x + 1)\left[ \left( \frac{2 - \frac{1}{8}x^4 + 30x^2}{(3x + 10x^3)(2x^4 - 6)} \right) + (x^{2/3} + 10x^3)(8x^3) \right]
\]
\[
\frac{\left[ (x^{2/3} + 10x^3)(2x^4 - 6) \right]^2}{\left[ (x^{2/3} + 10x^3)(2x^4 - 6) \right]^2}
\]