You have 50 minutes to complete this test. You must show all work to receive full credit. Work any 7 of the following 8 problems. Clearly cross out the problem you do not wish me to grade. Each problem is worth 14 points, and you get 2 points for free, for a total of 100 points. The answers will be posted on the electronic reserves later today.

1. Suppose \( f(x) = \frac{x^2}{x+2} \). Find all vertical and horizontal asymptotes (if none, say so), list the intervals where the function is increasing and where it is decreasing, and find all of the maximum and minimum points.

   **Vertical asymptote:** \( x = -2 \)

   **Horizontal asymptote:** none

   \[
   f'(x) = \frac{(2x)(x+2) - (x^2)(1)}{(x+2)^2} = \frac{2x^2 + 4x - x^2}{(x+2)^2} = \frac{x^2 + 4x}{(x+2)^2} = \frac{x(x+4)}{(x+2)^2}
   \]

   \( CN : x = -4, -2, 0 \)

   \( \begin{array}{c}
   0 \downarrow \downarrow \downarrow \uparrow \\
   -4 \downarrow \downarrow \downarrow \uparrow \\
   -2 \downarrow \downarrow \downarrow \uparrow \\
   0 \downarrow \downarrow \downarrow \uparrow \\
   \end{array} \)

   

   Increasing on \((-\infty, -4) \cup (0, \infty)\) max at \((-4, -8)\)

   Decreasing on \((-4, -2) \cup (-2, 0)\) min at \((0, 0)\)

2. For the function \( f(x) = x^4 - 2x^2 + 3 \), find all critical numbers. Then use the second derivative test to tell whether each critical number will result in a maximum or a minimum (you don’t need to find y-values).

   \[
   f'(x) = 4x^3 - 4x
   = 4x(x^2 - 1)
   = 4x(x+1)(x-1)
   \]

   \( CN : x = 0, -1, 1 \)

   \[
   f''(x) = 12x^2 - 4x
   \]

   \( f''(0) = -4 < 0 \) concave down \( \searrow \) so \( x = 0 \) gives a max

   \( f''(-1) = -8 > 0 \) concave up \( \nearrow \) so \( x = 1, x = -1 \) give minima.
3. Suppose \( q(p) = 500 - 2p \) units of a product are demanded when price is \( p \) dollars per unit.

(a) Calculate the elasticity of demand when \( p = 200 \). At this price, is the demand elastic or inelastic?

\[
E(p) = \frac{p}{q} \cdot \frac{q'}{q} = \frac{p}{500-2p} \cdot \frac{-2p}{-2} = \frac{-2p}{500-2p} = \frac{-2}{250-p}
\]

\[E(200) = \frac{200}{200-250} = \frac{200}{-50} = -4\] elasticity of demand

\[|E(200)| = |-4| = 4 > 1, \text{ so demand is elastic.}\]

(b) Write a sentence explaining the meaning of your answer in (a). Be as specific as possible.

"If price goes up 1%, demand will go down 4%.

(In this case, \( q(200) = 100 \), so if price goes from \( 200 \) to \( 203 \), demand will go from 100 to 96.)"

(c) Give an example of a product in this price range that might behave this way.

"Luxury" items that cost about \( \$200 \). Maybe jewelry, really good wine..."

4. Sketch a nice BIG graph of a function with all the properties listed below. Make sure your graph is clearly labeled.

(a) \( f'(x) > 0 \) when \( x < 1 \), but \( f'(x) \leq 0 \) otherwise increasing/decreasing

(b) \( f''(x) > 0 \) when \( x < -1 \) and when \( -1 < x < 0 \), but \( f''(x) \leq 0 \) otherwise come up/down

(c) \( f(x) \) is undefined when \( x = -1 \) hole or a sing-p

(d) \( \lim_{x \to -\infty} f(x) = -2 \) \( HA \) \( y = -2 \) on left
5. A retailer has purchased several cases of imported wine as an investment. As the wine ages, its value initially increases, but eventually the wine will pass its prime and its value will decrease. Suppose that $x$ years from now, the value of a case will be changing at a rate of $53 - 10x$ dollars per year. Suppose in addition that storage rates for the cases of wine stay the same at $3$ per case per year. When should the retailer sell the wine in order to make the most profit?

\[
\begin{align*}
\text{Profit} &= \text{value} - \text{storage cost} \\
\overset{p}{=} &= v - 3x \quad \text{(for one case)} \\
\overset{p'}{=} &= v' - 3 \quad v' \text{ is rate, value changes, } = 53 - 10x \\
\overset{p}{=} &= 53 - 10x - 3 = 50 - 10x = 10(5 - x) \\
\overset{\text{CN}}{=} &= x = 5. \\
\overset{\text{max}}{=} &= ? \\
\overset{1}{=} &= p'' = -10 \quad \text{or} \quad \overset{2}{=} + + \rightarrow \overset{\text{max}}{=} \\
\overset{p''(5)}{=} &= -10 \quad \text{came down,} \\
&= x = 5 \text{ years will maximize profit.}
\end{align*}
\]

6. Find the equation of the line tangent to the graph of $x^2y - 2xy^3 + 6 = 2x + 2y$ at the point $(0,3)$.

\[
\begin{align*}
(x^2)(y') - (2x)(y^3) + 6 &= 2x + 2y \\
2xy + x^2y' - (2y^3 + 2x(3y^2y')) &= 2 + 2y' \\
2(0)(3) + (0)(y') - (2(27) + 2(0)(3(9)y')) &= 2 + 2y' \\
-54 &= 2 + 2y' \\
-56 &= 2y' \\
-28 &= y' = m \\
\text{Line: } y - 3 &= -28(x - 0) \\
&= y = -28x + 3
\end{align*}
\]
7. Find the absolute minimum and absolute maximum points of \( f(x) = (x^2 - 4)^3 \) on the interval \([-3,1] \).

\[
\begin{align*}
 f'(x) &= 3(x^2-4)^2(2x) \\
 &= 3 \left[ (x+2)(x-2) \right]^2(2x) \\
\text{CN:} \quad x &= -2, 2, 0 \\
&< \text{not in interval.}
\end{align*}
\]

Check CN's and endpoints:

\[
\begin{align*}
 f(-2) &= 0 \\
 f(0) &= -64 \quad \leftarrow \text{abs. min at (0, -64)} \\
 f(1) &= -27 \\
 f(-3) &= -125 \quad \leftarrow \text{abs. max at (-3, 125)}
\end{align*}
\]

8. Find \( f'(x) \) for the following functions. DO NOT simplify!

(a) \( f(x) = \left( \frac{x+1}{x} \right)^2 - \frac{5}{\sqrt{3x}} = (x + x^{-1})^2 - \frac{5}{\sqrt{3}} x^{-3/2} \)

\[
\begin{align*}
 f'(x) &= 2(x + x^{-1}) (1 - x^{-2}) + \frac{5}{2\sqrt{3}} x^{-3/2}
\end{align*}
\]

(b) \( f(x) = \frac{(3x+1)^2}{(1-3x)^4} \)

\[
\begin{align*}
 f'(x) &= \frac{3(3x+1)^2(3)(1-3x)^4 - (3x+1)^3(4)(1-3x)^3(-3)}{(1-3x)^6}
\end{align*}
\]