You have 50 minutes to complete this test. You must show all work to receive full credit. Work any 7 of the following 8 problems. Clearly CROSS OUT the problem you do not wish me to grade. Each problem is worth 14 points, and you get 2 points for free, for a total of 100 points. The answers will be posted on the electronic reserves later today.

1. Suppose \( f(x) = \frac{x^2}{x-2} \). Find all critical numbers, list the intervals of increase and decrease, and tell whether each critical number will result in a maximum, a minimum, or neither. You do not need to find the \( y \)-values for the extrema.

\[
f'(x) = \frac{(2x)(x-2) - x^2}{(x-2)^2} = \frac{x^2 - 4x}{(x-2)^2} = \frac{x(x-4)}{(x-2)^2}
\]

Critical Numbers: \( x = 0, 2, 4 \)

\[
x = 0 \quad \text{gives a max}
\]

\[
x = 2 \quad \text{gives neither (flat spot)}
\]

\[
x = 4 \quad \text{gives a min}
\]

Increasing on \((-\infty, 0) \cup (4, \infty)\)

Decreasing on \((0, 2) \cup (2, 4)\)

2. For the following functions, find all horizontal and vertical asymptotes (remember that an asymptote is a LINE, not a number). If there are no asymptotes, say so.

(a) \( f(x) = \frac{2x^2 + 3x + 1}{3x^2 - 5x + 2} = \frac{(2x+1)(x+1)}{(3x-2)(x-1)} \)

\[
\text{VA}: \ x = \frac{2}{3} \\
x = 1 \\
\]

\[
\text{HA}: \ y = \frac{2}{3}
\]

(b) \( f(x) = \frac{x+2}{x^2-4} = \frac{x+2}{(x+2)(x-2)} \)

\[
\text{VA}: \ x = 2 \\
\text{HA}: \ y = 0
\]

(c) \( f(x) = x - \frac{1}{x} = \frac{x^2 - 1}{x} = \frac{(x+1)(x-1)}{x} \)

\[
\text{VA}: \ x = 0 \\
\text{HA}: \ \text{none}
\]
3. Suppose that $q(p) = 200 - 2p^2$ units of a product are demanded when the price is set at $p$ dollars per unit, assuming $0 \leq p \leq 250$.

a) Calculate the elasticity of demand when $p = 6$.

$$E(p) = \frac{p}{q} \cdot \frac{q'}{q} = \frac{p}{200 - 2p^2} \cdot ( -2p) = \frac{-4p^2}{200 - 2p^2} = \frac{-2p^2}{100 - p^2}$$

$$E(6) = \frac{-72}{100 - 36} = \frac{-72}{64} = \frac{-9}{8}$$

b) Is the demand for the product elastic or inelastic at $p = 6$?

$$|E(6)| = \left|\frac{-9}{8}\right| = \frac{9}{8} > 1$$, so demand is **elastic**.

c) Give an example of a product in the correct price range whose demand function would, in general, behave as in (a).

**Costs about $60, luxury...** (many choices)

4. Sketch a nice BIG graph of a function with all the properties listed below. Make sure your graph is clearly labeled.

a) $f''(x) < 0$ for $2 < x < 4$, but $f''(x) \geq 0$ otherwise

b) $f''(x) < 0$ for $x < 0$ and also for $x > 6$, but $f''(x) \geq 0$ otherwise

c) $f(x)$ is undefined when $x = 2$ **VA or hole**

d) $\lim_{x \to \infty} (x) = 3$. **HA** $y = 3$ (on right side)

\[y = 3\]

\[x = 2\]
5. Find \( f'(x) \) for the following functions. DO NOT simplify!

(a) \[ f(x) = x^2 (3 - 2x)^3 \]
\[ f'(x) = 2x (3 - 2x)^3 + x^2 (3)(3 - 2x)^2 (-2) \]

(b) \[ f(x) = \sqrt{\frac{1 - 2x}{3x - 2}} = \left( \frac{1 - 2x}{3x - 2} \right)^{\frac{1}{2}} \]
\[ f'(x) = \frac{1}{2} \left( \frac{1 - 2x}{3x - 2} \right)^{-\frac{1}{2}} \left( \frac{(-2)(3x - 2) - (1 - 2x)(3)}{(3x - 2)^2} \right) \]

6. Find the equation of the line tangent to the curve \((3xy^2 + 1)^4 = 2x - 3y\) at the point \(\left( \frac{1}{2}, 0 \right)\).

\[ 4 (3xy^2 + 1)^3 \left( 3y^2 + 6xyy' \right) = 2 - 3y' \]

Fill in \( x = \frac{1}{2}, y = 0 \)

\[ 4 \left( 0 + 1 \right)^3 (0 + 0) = 2 - 3y' \]
\[ 4 \cdot 1 = 2 - 3y' \]
\[ 0 = 2 - 3y' \]
\[ 3y' = 2 \]
\[ y' = \frac{2}{3} = m \]

\[ \text{Line: } y = \frac{2}{3} (x - \frac{1}{2}) \]
7. Find the absolute minimum and absolute maximum points of $f(x) = \frac{1}{3} x^3 - 9x + 2$ on the interval $0 \leq x \leq 2$.

$$f'(x) = x^2 - 9 = (x+3)(x-3)$$

CN: $x = 3, -3$. Notice these are both outside the interval. So we only need to check the endpoints.

$$f(0) = 2$$
$$f(2) = \frac{8}{3} - 18 + 2 = \frac{8}{3} - \frac{40}{3} = -\frac{40}{3}$$

absolute min $(2, -\frac{40}{3})$
absolute max $(0, 2)$

8. Mrs. Jones runs a small insurance company that sells policies for a large firm. Mrs. Jones does not sell policies herself, but she is paid a commission of $50 for each policy sold by her employees. When she employs $m$ salespeople, her company will sell $q$ policies each week, where $q = m^3 - 12m^2 + 60m$. She pays her employees $750 per week, and her weekly fixed costs are $2500$. Her office can accommodate at most 7 employees. How many employees should she have in order to maximize her weekly profit?

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

$$P = 50 \cdot \text{(number of policies)} - 750 \cdot \text{(number of employees)} - 2500$$

$$P = 50 \cdot (m^3 - 12m^2 + 60m) - 750m = 2500$$

$$P = 50m^3 - 600m^2 + 2250m - 2500$$

$$P' = 150m^2 - 1200m + 2250$$

$$P' = 150(m^2 - 8m + 15) = 150(m - 3)(m - 5)$$

CN: $m = 3, 5$

$$\left[ \begin{array}{c} + \hspace{0.5cm} - \hspace{0.5cm} + \end{array} \right] \quad P'$$

$$\text{max} \quad \text{min}$$

$P(3) = 200$  $\text{She should hire 7 employees.}$

$P(7) = 1000$