NAME KEY

Math 1212
Test 3
Spring 2016

You have 50 minutes to complete this test. You must *show all work* to receive full credit. Work any 6 of the following 7 problems. Clearly **CROSS OUT** the problem you do not wish me to grade. Each problem is worth 16 points, and you get 4 points for free, for a total of 100 points. The answers will be posted on the electronic reserves later today.

1. Solve $\dfrac{dy}{dx} = y^2\sqrt{4-x}$ if $y = 2$ when $x = 4$.

$$\frac{dy}{dx} = y^2(4-x)^{1/2}$$

$$y^{-2}\,dy = (4-x)^{1/2}\,dx$$

$$\int y^{-2}\,dy = \int (4-x)^{1/2}\,dx$$

$u = 4-x$
$du = -dx$
$-du = dx$

$$\frac{y^{-1}}{-1} = -\frac{2}{3}(4-x)^{3/2} + C$$

$$\frac{1}{y} = \frac{2}{3}(4-x)^{3/2} - C$$

If $x=4$, $y=2$, so

$$\frac{1}{2} = \frac{2}{3}(0)^{3/2} - C$$

$$-\frac{1}{2} = C$$

$$\frac{1}{y} = \frac{2}{3}(4-x)^{3/2} + \frac{1}{2}$$

$$y = \frac{1}{\frac{2}{3}(4-x)^{3/2} + \frac{1}{2}}.$$

2. Find $f'(x)$ for the following functions. DO NOT simplify!

(a) $f(x) = \dfrac{4e^{3x}}{x^2+1}$

$$f'(x) = \frac{(12e^{3x})(x^2+1) - (4e^{3x})(2x)}{(x^2+1)^2}$$

(b) $f(x) = \ln\sqrt{3x^2+4x-1} = \ln(3x^2+4x-1)^{1/2}$

$$f'(x) = \frac{1}{(3x^2+4x-1)^{1/2}}\left(\frac{1}{2}\right)(3x^2+4x-1)^{-1/2}(6x+4)$$

3. If you invest \$100,000 at 12% annual interest compounded continuously, how long will it take for you to become a millionaire?

$B = Pe^{rt}$

$1{,}000{,}000 = 100{,}000\, e^{0.12t}$

$10 = e^{0.12t}$

$\ln 10 = 0.12t$

$\frac{\ln 10}{0.12} = t \approx 19.188$ years

(19 years, 2 months, 1 week)

4. Suppose that the undergraduate enrollment at S&T grows exponentially. According to the enrollment count on the registrar's webpages, in the fall of 2005, our total enrollment was 5602. It grew to 8889 by the fall of 2015. In what year will our enrollment top 10000?

①	F05	$t=0$	$B=5602$
②	F15	$t=10$	$B=8889$
③		$t=?$	$B=10{,}000$

① $B = Pe^{rt}$

$5602 = Pe^{r(0)}$

$5602 = P$

so $B = 5602e^{rt}$.

② $8889 = 5602e^{r(10)}$

$\frac{8889}{5602} = e^{10r}$

$\ln\left(\frac{8889}{5602}\right) = 10r$

$\frac{\ln\left(\frac{8889}{5602}\right)}{10} = r \approx 0.0462$

So our equation is now

$B = 5602e^{0.0462t}$.

③ $10000 = 5602e^{0.0462t}$

$\frac{10000}{5602} = e^{0.0462t}$

$\ln\frac{10000}{5602} = 0.0462t$

$t = \frac{\ln\frac{10000}{5602}}{0.0462}$

$t \approx 12.54$ years

Enrollment will top 10,000 in Fall 2018.

5. a) If $\log_2 x = 2(\log_2 3 - \log_2 5)$, find x.

$$\log_2 x = 2\log_2\left(\tfrac{3}{5}\right)$$
$$\log_2 x = \log_2\left(\tfrac{9}{25}\right)$$
$$x = \frac{9}{25}$$

b) If $\ln a = -1$, $\ln b = 2$, and $\ln c = 4$, calculate $\ln\dfrac{a^3}{\sqrt{bc}}$.

$$\begin{aligned}\ln\frac{a^3}{\sqrt{bc}} &= \ln a^3 - \ln (bc)^{1/2} \\ &= 3\ln a - \tfrac{1}{2}(\ln b + \ln c) \\ &= 3(-1) - \tfrac{1}{2}(2+4) \\ &= -3 - \tfrac{1}{2}(6) = -3-3 = -6\end{aligned}$$

6. Evaluate the following integrals:

a) $\int (x^2-1)(x^3-3x)^4\,dx$

Let $u = x^3 - 3x$

Then $du = (3x^2-3)\,dx$

$\tfrac{1}{3}\,du = (x^2-1)\,dx$

$$\begin{aligned}&= \int u^4\left(\tfrac{1}{3}\,du\right) \\ &= \tfrac{1}{3}\int u^4\,du \\ &= \tfrac{1}{3}\left(\tfrac{1}{5}u^5\right) + C \\ &= \tfrac{1}{15}(x^3-3x)^5 + C\end{aligned}$$

b) $\int \dfrac{2x+3}{x^2}\,dx$

$$\begin{aligned}&= \int (2x^{-1} + 3x^{-2})\,dx \\ &= 2\ln|x| + 3\frac{x^{-1}}{-1} + C \\ &= 2\ln|x| - \frac{3}{x} + C\end{aligned}$$

c) $\int x^2 \ln x\,dx$

Let $u = \ln x$ $\quad dv = x^2\,dx$

$du = \frac{1}{x}\,dx$ $\quad v = \frac{1}{3}x^3$

$$\begin{aligned}= uv - \int v\,du &= (\ln x)\left(\tfrac{1}{3}x^3\right) - \int\left(\tfrac{1}{3}x^3\right)\left(\tfrac{1}{x}\,dx\right) \\ &= \tfrac{1}{3}x^3\ln x - \tfrac{1}{3}\int x^2\,dx \\ &= \tfrac{1}{3}x^3\ln x - \tfrac{1}{9}x^3 + C\end{aligned}$$

7. Find all maxima, minima and inflection points of $f(x) = xe^{-x}$. Also give the intervals where f is increasing, decreasing, concave up, and concave down. Then carefully sketch the graph of f, including all asymptotes. Be sure to label the asymptotes, extrema, and inflection points.

$$f'(x) = (1)(e^{-x}) + (x)(-e^{-x})$$
$$= e^{-x}(1-x) = 0$$

CN: $x = 1$

f': $+$ before 1, $-$ after 1; max at 1

$$f''(x) = (-e^{-x})(1-x) + (e^{-x})(-1)$$
$$= -e^{-x} + xe^{-x} - e^{-x}$$
$$= e^{-x}(-1 + x - 1)$$
$$= e^{-x}(x-2) = 0$$

IN: $x = 2$

f'': $-$ before 2, $+$ after 2

VA: None, since $f(x)$ is defined for all x, $y = \frac{x}{e^x}$ ← (never zero)

HA: If x gets big +, $y \to \frac{\text{big}+}{\text{really big}+} \to 0$

If x gets big −, $y \to \frac{\text{big}-}{\text{small}+} \to -\infty$

HA $y=0$ on right
Function gives big negatives on left

Results

increasing on $(-\infty, 1)$
decreasing on $(1, \infty)$
maximum $(1, \frac{1}{e}) \approx (1, 0.37)$
no minimum
conc up on $(2, \infty)$
conc down on $(-\infty, 2)$
inf. pt $(2, \frac{2}{e^2}) \approx (2, 0.27)$
HA: $y = 0$
VA: none

combo: inc down on $(-\infty, 1)$; dec down on $(1, 2)$; dec up on $(2, \infty)$