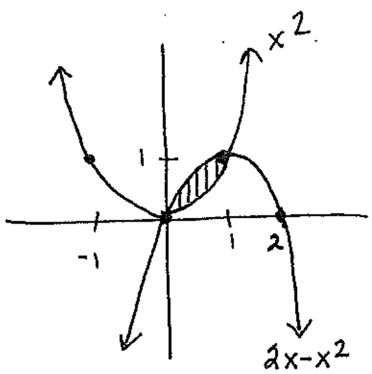
NAME KEY

Math 1212
Test 4
Spring 2015

You have 50 minutes to complete this test. You must *show all work* to receive full credit. Work any 6 of the following 7 problems. Clearly **CROSS OUT** the problem you do not wish me to grade. Each problem is worth 16 points, and you get 4 points for free, for a total of 100 points. The answers will be posted on the electronic reserves later today.

1. Find the area of the region bounded by the curves $y = x^2$ and $y = 2x - x^2$. Be sure to sketch a graph first!



Intersection Points: $x^2 = 2x - x^2$

$2x^2 - 2x = 0$

$2x(x-1) = 0$

$x = 0, 1$

Area: $A = \int_0^1 (\text{top} - \text{bottom})\, dx$

$= \int_0^1 (2x - x^2 - x^2)\, dx$

$= \int_0^1 (2x - 2x^2)\, dx$

$= \left[x^2 - \frac{2}{3}x^3\right]_0^1 = \left(1 - \frac{2}{3}\right) - (0-0)$

$= 1/3$

2. For $f(x,y,z) = x^2y^3 - \frac{x}{z} + e^x \ln y$, find f_x, f_y, and f_z.

$f(x,y,z) = x^2y^3 - xz^{-1} + e^x \ln y$

$f_x = 2xy^3 - z^{-1} + e^x \ln y$

$f_y = 3x^2y^2 + e^x\left(\frac{1}{y}\right)$

$f_z = xz^{-2}$

3. Find and classify the critical points of $f(x,y) = -x^4 + 4xy - 2y^2 + 1$.

$f_x = -4x^3 + 4y = 0 \longrightarrow -4x^3 + 4x = 0$

$f_y = 4x - 4y = 0 \longrightarrow x = y$

$-4x(x^2 - 1) = 0$

$-4x(x+1)(x-1) = 0$

$x = 0, 1, -1$

crit. pts. $(0,0), (1,1), (-1,-1)$

3

$f_{xx} = -12x^2$

$f_{yy} = -4$

$f_{xy} = 4$

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = 48x^2 - 16$

$D(0,0) = -16 < 0$, so $(0,0)$ gives a saddle point.

$D(1,1) = 48 - 16 > 0$, $f_{xx}(1,1) = -12 < 0$ (concave down)

so $(1,1)$ gives a maximum

$D(-1,-1) = 48 - 16 > 0$, $f_{xx}(-1,-1) = -12 < 0$

so $(-1,-1)$ gives a maximum.

4. Suppose p_1 and p_2 are the prices of two products. Also suppose $D_1(p_1, p_2) = 500 - 0.5p_1 - p_2^2$ and $D_2(p_1, p_2) = 10{,}000 - 8p_1 - 100p_2^2$ are the demand functions for the two products (quantities). Answer the following questions, showing your work below.

a) Are these two products competitive (substitutes), complementary, or neither?

$\frac{\partial D_1}{\partial p_2} = -2p_2 < 0$ since $p_2 > 0$

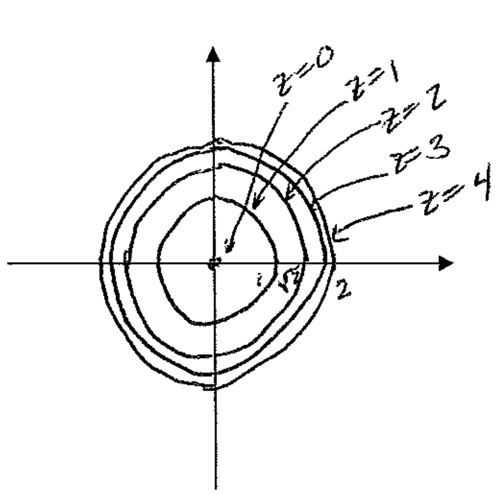
$\frac{\partial D_2}{\partial p_1} = -8 < 0$

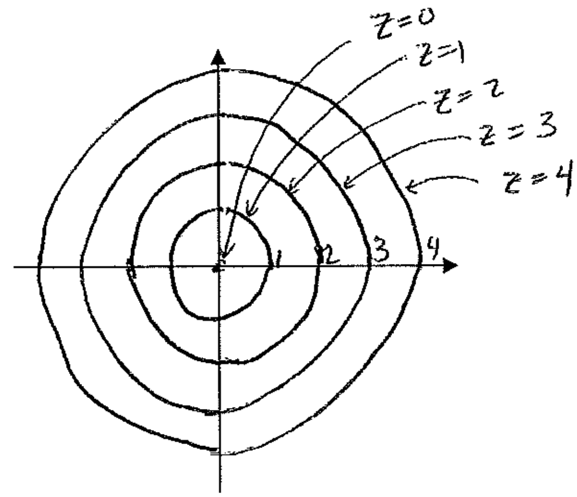
Both negative

Products are complementary

b) An example of two products that might behave this way are tennis racquets and tennis balls.

5. Sketch at least three level curves for $z = x^2 + y^2$ on the first set of axes below. Then sketch at least three level curves for $z^2 = x^2 + y^2$ on the second set of axes. Describe the three dimensional surface represented by each and why you think each surface has that shape based on your level curves.





$z = x^2 + y^2$

$z=0$: $0 = x^2+y^2$ point $(0,0)$

$z=1$: $1 = x^2+y^2$ radius 1

$z=2$: $2 = x^2+y^2$ radius $\sqrt{2}$

$z=3$: $3 = x^2+y^2$ radius $\sqrt{3}$

⋮

this is a paraboloid

sides get steeper as you go out

$z^2 = x^2 + y^2$

$z=0$: $0 = x^2+y^2$ point $(0,0)$

$z=1$: $1 = x^2+y^2$ radius 1

$z=2$: $4 = x^2+y^2$ radius 2

⋮

this is a cone, sides go up evenly

6. Calculate $\int_0^{\infty} \frac{x}{(x^2+5)^2}\,dx$.

$u = x^2+5$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$\int_0^{\infty} \frac{x}{(x^2+5)^2}\,dx = \lim_{n\to\infty} \int_0^n \frac{x}{(x^2+5)^2}\,dx$$

$$= \lim_{n\to\infty} \int_{x=0}^{x=n} \frac{1}{2}u^{-2}\,du$$

$$= \lim_{n\to\infty} \left[\frac{-1}{2}u^{-1}\right]_{x=0}^{x=n} = \lim_{n\to\infty}\left[\frac{-1}{2(x^2+5)}\right]_0^n$$

$$= \lim_{n\to\infty}\left[\cancel{\frac{-1}{2(n^2+5)}} + \frac{1}{2(5)}\right] = \frac{1}{10}$$

as $n\to\infty$, denom gets large, fraction $\to 0$.

7. Suppose a manufacturing firm has budgeted \$60,000 per month for labor and materials. If \$$x$ thousand is spent on labor and \$$y$ thousand is spent on materials, and if the monthly output (in units) is given by $N(x,y) = 4xy - 8x$, how should the budget be allocated in order to maximize the output N? What is the maximum output?

$x + y = 60$ (constraint).

maximize $N = 4xy - 8x$.

$$F(x,y,\lambda) = 4xy - 8x - \lambda(x+y-60)$$

$F_x = 4y - 8 - \lambda = 0 \longrightarrow 4y = \lambda + 8$, $\quad y = \dfrac{\lambda+8}{4}$

$F_y = 4x - \lambda = 0 \longrightarrow 4x = \lambda$, $\quad x = \dfrac{\lambda}{4}$

$F_\lambda = -x - y + 60 = 0 \longrightarrow -\dfrac{\lambda}{4} - \dfrac{\lambda+8}{4} + 60 = 0$

$$60 = \frac{\lambda + (\lambda+8)}{4}$$

$240 = 2\lambda + 8$

$232 = 2\lambda$

$116 = \lambda$

$x = 29$

$y = \dfrac{116+8}{4} = 31$

Allocate \$29,000 to labor and \$31,000 to materials.
The output will then be maximized at

$$N(29,31) = 4(29)(31) - 8(29)$$
$$= 3596 - 232$$
$$= 3364 \text{ units}$$