Section 5.3 - Introduction to Differential Equations

2.
$$\frac{dP}{dt} = \sqrt{t} + e^{-t}$$
$$dP = (t^{1/2} + e^{-t})\,dt$$
$$\int dP = \int (t^{1/2} + e^{-t})\,dt$$
$$P = \frac{2}{3} t^{3/2} - e^{-t} + C$$

6.
$$\frac{dy}{dx} = e^{x+y} = e^x e^y$$
$$e^{-y}\,dy = e^x\,dx$$
$$\int e^{-y}\,dy = \int e^x\,dx$$
$$-e^{-y} = e^x + C$$
$$e^{-y} = -e^x - C$$
$$e^y = \frac{1}{-e^x - C}$$
$$y = \ln\left(\frac{1}{-e^x - C}\right)$$

10.
$$\frac{dy}{dx} = \frac{y^2+4}{xy}$$
$$\frac{y\,dy}{y^2+4} = \frac{dx}{x}$$
$$\int \frac{y\,dy}{y^2+4} = \int \frac{1}{x}\,dx$$
$$u = y^2 + 4$$
$$du = 2y\,dy$$
$$\frac{1}{2}\,du = y\,dy$$
$$\frac{1}{2}\int \frac{1}{u}\,du = \int \frac{1}{x}\,dx$$
$$\frac{1}{2}\ln|y^2+4| = \ln|x| + C$$
$$\ln(y^2+4) = 2\ln|x| + 2C$$
$$= \ln x^2 + 2C$$
$$y^2 + 4 = e^{\ln x^2 + 2C} = x^2 e^{2C} = kx^2$$
$$y^2 = kx^2 - 4$$
$$y = \pm\sqrt{kx^2 - 4}$$

Section 5.3 - (cont)

14. $\frac{dy}{dx} = (e^y + 1)(x-2)^9$

$\frac{dy}{e^y + 1} = (x-2)^9 dx$

↑ this one is too hard !!

18. $\frac{dy}{dx} = 5x^4 - 3x^2 - 2$, $y = 4$ when $x = 1$.

$y = \int (5x^4 - 3x^2 - 2)\,dx$

$= x^5 - x^3 - 2x + C$

$4 = 1 - 1 - 2 + C$

$C = 6$

$y = x^5 - x^3 - 2x + 6$

22. $\frac{dy}{dx} = x e^{y - x^2}$, $y = 0$ when $x = 1$.

$= x e^y e^{-x^2}$

$e^{-y} dy = x e^{-x^2} dx$

$\int e^{-y} dy = \int x e^{-x^2} dx$

$-e^{-y} = \int x e^{-x^2} dx$ $\qquad u = -x^2$

$= -\frac{1}{2} \int e^u \, du$ $\qquad du = -2x\,dx$

$= -\frac{1}{2} e^{-x^2} + C$ $\qquad -\frac{1}{2} du = x\,dx$

$-e^0 = -\frac{1}{2} e^{-1} + C$

$-1 = -\frac{1}{2e} + C$

$C = -1 + \frac{1}{2e}$

$-e^{-y} = -\frac{1}{2} e^{-x^2} + \frac{1}{2e} - 1$

$e^{-y} = \frac{1}{2} e^{-x^2} - \frac{1}{2e} + 1$

$e^y = \frac{1}{\frac{1}{2} e^{-x^2} - \frac{1}{2e} + 1}$

$y = \ln\left(\frac{1}{\frac{1}{2} e^{-x^2} - \frac{1}{2e} + 1}\right)$

Section 5.3 - (cont)

28. A = amount of radium at time t.

$\frac{dA}{dt} = kA$ (k is a negative proportionality constant since A is decaying).

29. B = amount the money is worth at time t.

$\frac{dB}{dt} = 0.07B$

31. P = population of town at time t.

$\frac{dP}{dt} = 500$ (change in population is 500 people per year).

32. T = total facts in memory

F = # facts recalled at time t

$T - F$ = # facts not recalled by time t.

$\frac{dF}{dt} = k(T-F)$.

37. $D(p) = a - bp$, $S(p) = r + sp$.

$$\frac{dp}{dt} = k(D-S) = k(a-bp-r-sp).$$

a, b, r, s, k are constant.

$$= k(a-r) - k(b+s)p$$

Let $A = k(a-r)$, $B = k(b+s)$

$$= A - Bp$$

$$\frac{dp}{A-Bp} = dt$$

$$\int \frac{dp}{A-Bp} = \int dt = t + C_1$$

$u = A - Bp$

$du = -B\,dp$

$-\frac{1}{B}\,du = dp$

$$-\frac{1}{B}\int \frac{du}{u} = t + C_1$$

$$-\frac{1}{B}\ln|A-Bp| = t + C_1$$

$$A - Bp = e^{-Bt-BC_1}$$

$$-Bp = -A + e^{-B(t+C_1)}$$

(over →)

Section 5.3 - cont

37. (cont)

$$P = \frac{A - e^{-B(t+c_1)}}{B} = \frac{K(a-r) - e^{-K(b+s)(t+c_1)}}{K(b+s)}$$

$$= \frac{K(a-r) - e^{-Kt(b+s)} e^{-Kc_1(b+s)}}{K(b+s)}$$ all this is constant (there are no t's) so call it c_2.

$$= \frac{\cancel{K}(a-r) - \cancel{c_2} e^{-kt(b+s)}}{\cancel{K}(b+s)}$$ Now let $C = c_2/K$, so $c_2 = KC$

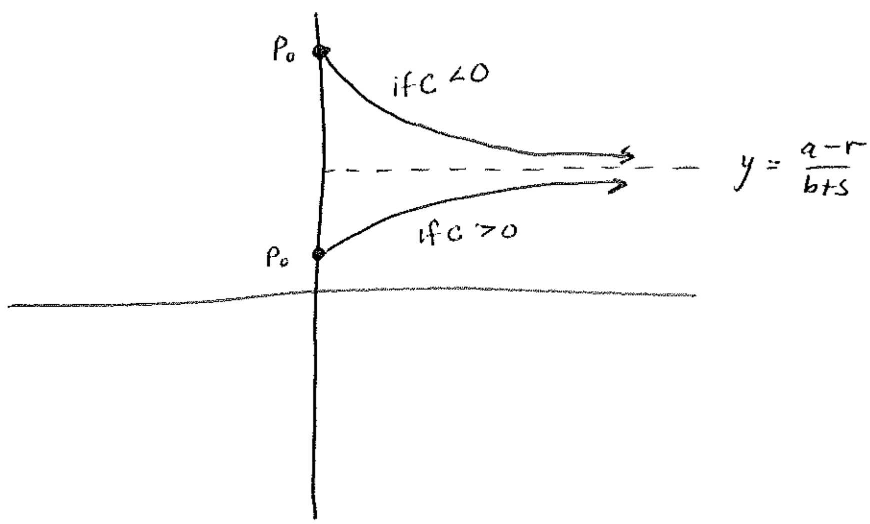
$$= \frac{(a-r) - Ce^{-kt(b+s)}}{b+s}$$

Notice if $t=0$, $P_0 = \frac{a-r-c}{b+s}$. If $c>0$, $\frac{a-r-c}{b+s} < \frac{a-r}{b+s}$

If $c<0$, $\frac{a-r-c}{b+s} > \frac{a-r}{b+s}$

As $t \to \infty$, $P \to \frac{a-r}{b+s}$, so there is a horizontal asymp at $y = \frac{a-r}{b+s}$.

Graph:



If $D > S$, there is a shortage, and $a - bp > r + sp \cdots p < \frac{a-r}{b+s}$ (graph is the bottom one)

If $D < S$, there is a surplus, and $a - bp < r + sp \cdots p > \frac{a-r}{b+s}$ (graph is the top one)

Section 5.4 – Integration by Parts

2. $\int x e^{x/2}\,dx \qquad u = x \qquad dv = e^{x/2}\,dx$

$du = dx \qquad v = \int e^{x/2}\,dx = 2e^{x/2}$

$= 2xe^{x/2} - 2\int e^{x/2}\,dx$

$= 2xe^{x/2} - 4e^{x/2} + C$

6. $\int t \ln t^2\,dt \qquad u = \ln t^2 \qquad dv = t\,dt$

$du = \frac{1}{t^2}(2t)\,dt \qquad v = \frac{1}{2}t^2$

$= \frac{1}{2}t^2 \ln t^2 - \int \frac{1}{2}t^2 \left(\frac{1}{t^2}\right)(2t)\,dt$

$= \frac{1}{2}t^2 \ln t^2 - \int t\,dt$

$= \frac{1}{2}t^2 \ln t^2 - \frac{1}{2}t^2 + C$

10. $\int x\sqrt{1-x}\,dx \qquad u = x \qquad dv = (1-x)^{1/2}\,dx$

$du = dx \qquad v = -\frac{2}{3}(1-x)^{3/2}$

$= -\frac{2}{3}x(1-x)^{3/2} + \frac{2}{3}\int (1-x)^{3/2}\,dx$

$= -\frac{2}{3}x(1-x)^{3/2} + \frac{2}{3}\cdot -\frac{2}{5}(1-x)^{5/2} + C$

$= -\frac{2}{3}x(1-x)^{3/2} - \frac{4}{15}(1-x)^{5/2} + C$

14. $\int \frac{x}{\sqrt{2x+1}}\,dx \qquad u = x \qquad dv = (2x+1)^{-1/2}\,dx$

$du = dx \qquad v = (2x+1)^{1/2}$

$= x(2x+1)^{1/2} - \int (2x+1)^{1/2}\,dx$

$= x(2x+1)^{1/2} - \frac{2}{3}\left(\frac{1}{2}\right)(2x+1)^{3/2} + C$

$= x(2x+1)^{1/2} - \frac{1}{3}(2x+1)^{3/2} + C$

Section 5.4- (cont)

18. $\int x^3 e^{2x}\,dx$ $\qquad u = x^3 \qquad dv = e^{2x}\,dx$

$\qquad du = 3x^2 dx \qquad v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}x^3e^{2x} - \frac{3}{2}\int x^2 e^{2x}\,dx$

$\qquad u = x^2 \qquad dv = e^{2x}dx$

$\qquad du = 2x\,dx \qquad v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}x^3e^{2x} - \frac{3}{2}\left[\frac{1}{2}x^2e^{2x} - \int xe^{2x}\,dx\right]$

$= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{2}\int xe^{2x}\,dx$

$\qquad u = x \qquad dv = e^{2x}dx$

$\qquad du = dx \qquad v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{2}\left[\frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx\right]$

$= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{1}{4}e^{2x} + C$

22. $\int \frac{\ln x}{x^3}\,dx$ $\qquad u = \ln x \qquad dv = x^{-3}dx$

$\qquad du = \frac{1}{x}\,dx \qquad v = \frac{-1}{2}x^{-2}$

$= \frac{-1}{2}x^{-2}\ln x + \frac{1}{2}\int x^{-3}\,dx$

$= \frac{-1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2} + C$

26. $\frac{dy}{dx} = (x+1)e^{-x}$, through $(1,5)$.

$y = \int (x+1)e^{-x}\,dx$ $\qquad u = x+1 \qquad dv = e^{-x}dx$

$\qquad du = dx \qquad v = -e^{-x}$

$= -(x+1)e^{-x} + \int e^{-x}\,dx$

$= -(x+1)e^{-x} - e^{-x} + C.$

$5 = -2e^{-1} - e^{-1} + C = \frac{-3}{e} + C$

$C = 5 + 3/e.$

$y = -(x+1)e^{-x} - e^{-x} + 5 + 3/e$

Section 5.4 - (cont).

30. $\frac{dy}{dt} = 2000\, t\, e^{-0.2t}$ dollars per week. When $t=0$, $y=0$, since no money was raised before the start of the campaign.

$y = 2000 \int t e^{-0.2t}\, dt.$ $\quad u = t \quad dv = e^{-0.2t}dt$

$\quad du = dt \quad v = -5\, e^{-0.2t}$

$= 2000 \left[-5 t e^{-0.2t} + 5 \int e^{-0.2t} dt \right]$

$= -10{,}000\, t e^{-0.2t} + 10000 \left(-5 e^{-0.2t} \right) + C$

$0 = 0 - 50000 + C$, so $C = 50000$

$y = -10000\, t e^{-0.2t} - 50000\, e^{-0.2t} + 50000$

During the first 5 weeks,

$y = -50000\, e^{-1} - 50000\, e^{-1} + 50000$

$= 50000 - \frac{100000}{e}$

$\approx \$13{,}212.06$ was raised.