You have 50 minutes to complete this test. You must show all work to receive full credit. Work any 7 of the following 8 problems. Clearly cross out the problem you do not wish me to grade. Each problem is worth 14 points, and you get 2 points for free, for a total of 100 points. The answers will be posted on the electronic reserves later today.

1. Given the function $f(x) = x^4 - 5x$, list the intervals where $f$ is increasing, where it is decreasing, where it is concave up and where it is concave down. Find all extrema and inflection points. Do not sketch the graph.

\[
\begin{align*}
\frac{f'(x)}{} &= 5x^4 - 5 \\
&= 5(x^4 - 1) \\
&= 5((x^2 - 1)(x^2 + 1)) \\
&= 5(x - 1)(x + 1)(x^2 + 1) \\
\frac{f''(x)}{} &= 20x^3 \\
\end{align*}
\]

Increasing on $(-\infty, -1) \cup (1, \infty)$
Decreasing on $(-1, 1)$
Max $(-1, 4)$
Min $(1, -4)$
Conc up $(0, \infty)$
Conc down $(-\infty, 0)$
Infl. pt $(0,0)$

2. For the following functions, find all horizontal and vertical asymptotes (remember that an asymptote is a line, not a number). If there are no asymptotes, say so.

a)

\[
f(x) = \frac{-3x^2}{x^2 - 16} = \frac{3x^2}{(x+4)(x-4)}
\]

VA: $x = 4, x = -4$
HA: $y = 3$

b)

\[
f(x) = \frac{x^2 - 2x - 3}{x^3 - 5x^2 + 6x} = \frac{(x - 3)(x + 1)}{x(x - 3)(x - 2)}
\]

VA: $x = 0, x = 2$
HA: $y = 0$

(cross out)

HA: $x = 3$


b) $f(x) = \frac{x^2 - 2x - 3}{x^3 - 5x^2 + 6x} = \frac{(x - 3)(x + 1)}{x(x - 3)(x - 2)}$

VA: $x = 0, x = 2$
HA: $y = 0$

HA: $x = 3$

HA: $y = 0$


(cross out)


b) $f(x) = \frac{x^2 - 2x - 3}{x^3 - 5x^2 + 6x} = \frac{(x - 3)(x + 1)}{x(x - 3)(x - 2)}$

VA: $x = 0, x = 2$
HA: $y = 0$

HA: $x = 3$

HA: $y = 0$


(cross out)


b) $f(x) = \frac{x^2 - 2x - 3}{x^3 - 5x^2 + 6x} = \frac{(x - 3)(x + 1)}{x(x - 3)(x - 2)}$

VA: $x = 0, x = 2$
HA: $y = 0$

HA: $x = 3$

HA: $y = 0$


(cross out)


c) $f(x) = \frac{4}{x^2 + 1} + 7 = \frac{4 + 7(x^2 + 1)}{x^2 + 1}$

\[
= \frac{7x^2 + 11}{x^2 + 1}
\]

VA: None
HA: $y = 7$
3. Suppose that at price $p$, demand for a certain product is given by
   \[ q(p) = 10,000 - 500p \]
   when price is a positive value less than $20.
   
   a) Find the price elasticity of demand when price is $5$. Is demand elastic or
      inelastic at this price?
   \[
   E(p) = \frac{p}{q} \cdot \frac{q'}{q} = \frac{5}{10,000 - 500 \cdot (-500)} = \frac{1}{3} < 1,
   \]
   \[ E(5) = \frac{-2500}{10,000 - 2500} = -\frac{1}{3} \]
   demand is \underline{inelastic}
   
   b) Suppose the price of $5$ is increased to $5.15$. What is the percentage increase in
      price?
   \[
   \% \text{ increase} = \frac{\text{amt of increase}}{\text{orig amt}} \cdot 100\% = \frac{0.15}{5.00} \cdot 100\% = 3\%
   \]
   
   c) If price is increased from $5$ to $5.15$, use (a) and (b) to determine the new
      demand amount.
   by (a), if price ↑ 17%, demand ↓ \frac{1}{3} %
   using (b), if price ↑ 3%, demand ↓ 17%.
   old demand = \[ q(5) = 7500. \] 17% of this is 75, so
   new demand = \[ 7500 - 75 = 7425. \]

4. For the function \[ f(x) = \frac{4x}{x^2 + 1}, \text{ list the intervals where } f \text{ is concave up and where it is concave down. Find all inflection points. Find all asymptotes. If any of these items do not exist, say so.}
   
   \[ f'(x) = \frac{4(x^2+1)-(4x)(2x)}{(x^2+1)^2} = \frac{4x^2+4-8x^2}{(x^2+1)^2} = \frac{4-4x^2}{(x^2+1)^2} \]
   \[ f''(x) = \frac{(8x)(x^2+1)^2-(4-4x^2)(2)(x^2+1)(2x)}{(x^2+1)^4} \]
   \[ = \frac{-8x(x^2+1) - (4-4x^2)(4x)}{(x^2+1)^3} = \frac{-8x^3-8x-16x + 16x^3}{(x^2+1)^3} \]
   \[ = \frac{8x(x^2-3)}{(x^2+1)^3} \]
   \[ \text{IN: } x = 0, \pm \sqrt{3} \]
   
   \[ \begin{array}{cccc}
   \circ & + & + & + \\
   \end{array} \rightarrow f'' \]
   conc up on \((-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)\)
   conc down on \((-\infty, -\sqrt{3}) \cup (0, \sqrt{3})\)
   inf. pts \((-\sqrt{3}, -\sqrt{3}), (0, 0), (\sqrt{3}, \sqrt{3})\)
   VA: none
   HA: \ y = 0
5. Find the absolute extrema of $f(x) = x^4 - 8x^2 + 16$ on the interval $[1,3]$.

$$f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x + 2)(x - 2)$$

CN: $x = 2, -2, 2$ (not in interval)

$f(0) = 16$
$f(2) = 16 - 32 + 16 = 0 \rightarrow$ abs min $(2, 0)$
$f(1) = 1 - 8 + 16 = 9$
$f(3) = 81 - 72 + 16 = 25 \rightarrow$ abs max $(3, 25)$

6. Sketch the graph of a function $f(x)$ so that all conditions below are satisfied. Be sure your graph is big enough so I can see it and it is properly labeled.

a) $f(x)$ is defined for all $x$ except $x = 1$. $\rightarrow$ a symp or hole at $x = 1$

b) $f'(x) > 0$ when $1 < x < 3$. inc

c) $f''(x) > 0$ when $x > 4$ and when $x < -1$. up

d) $f(x)$ has an inflection point at $(-1, 2)$.

e) $\lim_{x \to \infty} f(x) = 2$. $\rightarrow$ a symp $y = 2$

\[\begin{array}{c}
\text{dec} & \text{dec} & \text{inc} & \text{dec} & \text{dec} \\
\text{up} & \text{down} & \text{down} & \text{down} & \text{up}
\end{array}\]
7. Find the equation of the line tangent to \(x^4 - 2x^3 - y^3 = 1 - y^3\) at the point \(2, 1\).

\[
4x^3 - (6x^2y^2 + 4x^3yy') = -3y^2y'
\]

Fill in \(x = 2, y = 1\)

\[
32 - (24 + 32y') = -3y'
\]

\[
32 - 24 = 32y' - 3y' = 8 = 29y' \\
y' = m = \frac{8}{29}
\]

\[
\text{Line: } y - 1 = \frac{8}{29} (x - 2)
\]

8. A company that makes iPad keyboard cases finds that if \(q\) packages of cases are produced in an hour, the average cost per package is \(\bar{c} = 2q^2 - 36q + 210 - \frac{200}{q}\), where the number of packages that can be produced per hour is restricted to \(2 \leq q \leq 10\). How many packages should be produced (within the restrictions) in order to maximize total cost? What is the minimum total cost in an hour?

\[
\text{Total cost} = C = \bar{c} \cdot q = 2q^3 - 36q^2 + 210q - 200
\]

\[
C' = 6q^2 - 72q + 210
\]

\[
= 6(q^2 - 12q + 35) = 6(q - 5)(q - 7)
\]

\[
\text{CN: } q = 5, 7
\]

\[
C(2) = 16 - 144 + 420 - 200 = 92 \quad \rightarrow \quad \text{produce 2 plgs per hour at a minimum total cost of $92 per hour}
\]

\[
C(5) = 250 - 900 + 1050 - 200 = 200
\]

\[
C(7) = 686 - 1764 + 1470 - 200 = 192
\]

\[
C(10) = 2000 - 3600 + 2100 - 200 = 300
\]

\[
C(5) = 250 - 900 + 1050 - 200 = 200
\]