You have 50 minutes to complete this test. You must show all work to receive full credit. Work any 6 of the following 7 problems. Clearly cross out the problem you do not wish me to grade. Each problem is worth 16 points, and you get 4 points for free, for a total of 100 points. The answers will be posted on the electronic reserves later today.

1. Find the area of the region bounded by $y = x^3$ and $y = 9x$. Be sure to sketch a graph first!

   \[
   \text{Intersection points:} \quad x^3 = 9x \\
   x^3 - 9x = 0 \\
   x(x^2 - 9) = 0 \\
   x = 0, \pm 3
   \]

   \[
   A = \int_{-3}^{0} (x^2 - 9x) \, dx + \int_{0}^{3} (9x - x^3) \, dx
   \]

   \[
   = \left[ \frac{1}{4} x^4 - \frac{9}{2} x^2 \right]_{-3}^{0} + \left[ \frac{9}{2} x^2 - \frac{1}{4} x^4 \right]_{0}^{3}
   \]

   \[
   = \left[ 0 - \left( \frac{81}{4} - \frac{81}{2} \right) \right] + \left[ \frac{81}{2} - \frac{81}{4} \right] - 0
   \]

   \[
   = \frac{81}{4} + \frac{81}{4} = \frac{81}{2}
   \]

2. Find all four second-order partial derivatives of $f(x, y) = e^{x^2y}$. Do NOT simplify.

   \[
   f_x = e^{x^2y} \cdot 2xy = 2xy e^{x^2y} \\
   f_y = e^{x^2y} \cdot x^2 = x^2 e^{x^2y}
   \]

   \[
   f_{xx} = 2y e^{x^2y} + 2xy e^{x^2y} \cdot 2xy \\
   f_{xy} = 2x e^{x^2y} + 2xy e^{x^2y} \cdot x^2 \\
   f_{yy} = x^2 e^{x^2y} \cdot x^2 \\
   f_{yx} = 2x e^{x^2y} + x^2 e^{x^2y} \cdot 2xy 
   \]

   equal, good!
3. Find and classify the critical points of \( f(x, y) = \frac{1}{3} x^3 + y^2 - 2x + 2y - 2xy \).

\[
\begin{align*}
\frac{\partial f}{\partial x} &= x^2 - 2 - 2y = 0 \\
\frac{\partial f}{\partial y} &= 2y + 2 - 2x = 0 \rightarrow x = y + 1 \\
(\frac{\partial^2 f}{\partial x^2}) &= 2x \\
(\frac{\partial^2 f}{\partial y^2}) &= 2 \\
(\frac{\partial^2 f}{\partial x \partial y}) &= -2 \\
D(x, y) &= \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - (\frac{\partial^2 f}{\partial x \partial y})^2 = 4x - 4
\end{align*}
\]

Critical points: \((2, 1), (0, -1)\)

\( x = 2, x = 0 \)

4. Suppose two products have demand equations \( D_1 = \frac{100}{p_1 \sqrt{p_2}} \) and \( D_2 = \frac{500}{p_2 \sqrt{p_1}} \), where \( p_1 \) and \( p_2 \) are the respective prices of the products. Are the products competitive, complementary, or neither? Give an example of two products that might behave this way.

\[
\frac{\partial D_1}{\partial p_2} = -50 p_1^{-1} p_2^{-3/2} < 0 \quad D_1 = 100 p_1^{-1} p_2^{-1/2}
\]

\[
\frac{\partial D_2}{\partial p_1} = -750 p_2^{-1} p_1^{-5/2} < 0 \quad D_2 = 500 p_2^{-1} p_1^{-3/2}
\]

The products are complementary.

Examples: hot dogs & hot dog buns, Corona & Lime...
5. Using four rectangles, estimate the area under the curve \( y = 10x - x^2 \) between \( x = 1 \) and \( x = 7 \). Then find the exact area.

\[
Y = -(x^2 - 10x) = -x(x-10) \quad \text{parabola, opens down, (0,0) & (10,0)}
\]

**Estimate**

\[
A \approx R_1 + R_2 + R_3 + R_4
\]

\[
\approx \frac{3}{2} \left( f\left(\frac{5}{2}\right) + \frac{3}{2} f\left(4\right) + \frac{3}{2} f\left(\frac{11}{2}\right) + \frac{3}{2} f(7) \right)
\]

\[
\approx \frac{3}{2} \left( 25 - \frac{25}{4} \right) + \frac{3}{2} (40 - 16) + \frac{3}{2} (55 - \frac{121}{4}) + \frac{3}{2} (70 - 49)
\]

\[
\approx \frac{3}{2} \left( \frac{75}{4} + 24 + \frac{99}{4} + 21 \right)
\]

\[
\approx \frac{3}{2} \left( \frac{87}{2} + 45 \right) \approx \frac{3}{2} \left( \frac{127}{2} \right) = \frac{631}{4}
\]

**Exact**

\[
A = \int_{1}^{7} (10x - x^2) \, dx = 5x^2 - \frac{x^3}{3} \bigg|_{1}^{7}
\]

\[
= (5(49) - \frac{343}{3}) - (5 - \frac{1}{3})
\]

\[
= 240 - \frac{342}{3} = 240 - 114 = 126
\]

6. Calculate \( \int_{1}^{\infty} \frac{x^2}{(x^3 + 2)^2} \, dx \).

\[
= \lim_{n \to \infty} \int_{1}^{n} \frac{x^2}{(x^3 + 2)^2} \, dx = \lim_{n \to \infty} \int_{1}^{x=n} \frac{1}{3} u^{-2} \, du = \lim_{n \to \infty} \left[ \frac{-1}{3} u^{-1} \right]_{x=1}^{x=n}
\]

\[
= \frac{-1}{3(x^3 + 2)} \bigg|_{1}^{n}
\]

\[
= \lim_{n \to \infty} \left[ \frac{-1}{3(n^3 + 2)} + \frac{1}{3(1^2)} \right]
\]

\[
= \frac{1}{9}
\]
A computer company has a monthly advertising budget of $60,000. Its marketing department estimates that if \( x \) dollars are spent each month on advertising in electronic media and \( y \) dollars per month are spent on television advertising, then the monthly sales will be \( S = 90x^{1/4}y^{3/4} \) dollars. If profit is 10% of sales, less the advertising cost, determine how to allocate the advertising budget in order to maximize monthly profit.

\[
x + y = 60000 \quad \text{constraint}
\]

\[
\text{Profit} = 0.10 (\text{sales}) - \text{adv. cost}
\]

\[
P = 0.10 (90x^{1/4}y^{3/4}) - 60000
\]

\[
P = 9x^{1/4}y^{3/4} - 60000 \quad \text{optimize this.}
\]

\[
F(x, y, \lambda) = 9x^{1/4}y^{3/4} - 60000 - \lambda (x + y - 60000)
\]

\[
= 9x^{1/4}y^{3/4} - 60000 - \lambda x - \lambda y + 60000\lambda
\]

\[
F_x = \frac{9}{4}x^{-3/4}y^{3/4} - \lambda = 0 \quad \rightarrow \quad \lambda = \frac{9y^{3/4}}{4x^{3/4}} = \frac{27x^{1/4}}{4y^{1/4}}
\]

\[
F_y = \frac{27}{4}x^{1/4}y^{-1/4} - \lambda = 0
\]

\[
F_\lambda = -x - y + 60000 = 0
\]

\[
-x - 3x + 60000 = 0
\]

\[
60000 = 4x
\]

\[
\begin{cases}
  x = 15000 \quad \text{on electronic media} \\
  y = 45000 \quad \text{on TV}
\end{cases}
\]