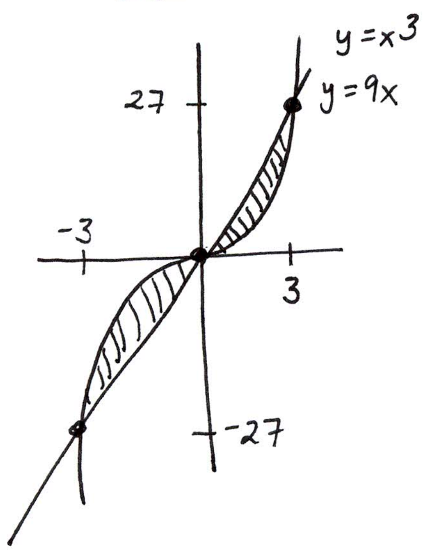
NAME KEY

Math 12
Test 4
Fall 2012

You have 50 minutes to complete this test. You must *show all work* to receive full credit. Work any 6 of the following 7 problems. Clearly **CROSS OUT** the problem you do not wish me to grade. Each problem is worth 16 points, and you get 4 points for free, for a total of 100 points. The answers will be posted on the electronic reserves later today.

1. Find the area of the region bounded by $y = x^3$ and $y = 9x$. Be sure to sketch a graph first!



Intersection points: $x^3 = 9x$
$x^3 - 9x = 0$
$x(x^2-9) = 0 \quad x = 0, \pm 3$

$$A = \int_{-3}^{0} (x^3 - 9x)\,dx + \int_0^3 (9x - x^3)\,dx$$

$$= \left[\frac{1}{4}x^4 - \frac{9}{2}x^2\right]_{-3}^{0} + \left[\frac{9}{2}x^2 - \frac{1}{4}x^4\right]_0^3$$

$$= \left[0 - \left(\frac{81}{4} - \frac{81}{2}\right)\right] + \left[\left(\frac{81}{2} - \frac{81}{4}\right) - 0\right]$$

$$= \frac{81}{4} + \frac{81}{4} = \frac{81}{2}$$

2. Find all four second-order partial derivatives of $f(x,y) = e^{x^2 y}$. Do NOT simplify.

$f_x = e^{x^2y} \cdot 2xy = 2xye^{x^2y}$
$f_y = e^{x^2y} \cdot x^2 = x^2 e^{x^2y}$

$f_{xx} = 2y\,e^{x^2y} + 2xy\,e^{x^2y} \cdot 2xy$
$f_{xy} = 2x\,e^{x^2y} + 2xye^{x^2y} \cdot x^2$
$f_{yy} = x^2 e^{x^2y} \cdot x^2$
$f_{yx} = 2xe^{x^2y} + x^2e^{x^2y} \cdot 2xy$

equal, good!

3. Find and classify the critical points of $f(x,y)=\frac{1}{3}x^3+y^2-2x+2y-2xy$.

$f_x = x^2-2-2y = 0$ ——— $(y^2+2y+1)-2-2y=0$

$f_y = 2y+2-2x = 0 \rightarrow x=y+1$

$y^2-1=0$

$y=\pm 1$

$x=2,\ x=0$

Critical points: $(2,1),\ (0,-1)$

$$\left.\begin{aligned} f_{xx} &= 2x \\ f_{yy} &= 2 \\ f_{xy} &= -2 \end{aligned}\right\} \quad D(x,y)=f_{xx}f_{yy}-(f_{xy})^2 = 4x-4$$

$D(2,1)=4(2)-4>0$, $f_{xx}(2,1)=2(2)>0$, minimum at $(2,1)$

$D(0,-1)=4(0)-4<0$, saddle point at $(0,-1)$.

4. Suppose two products have demand equations $D_1=\dfrac{100}{p_1\sqrt{p_2}}$ and $D_2=\dfrac{500}{p_2\sqrt[3]{p_1}}$, where p_1 and p_2 are the respective prices of the products. Are the products competitive, complementary, or neither? Give an example of two products that might behave this way.

$D_1 = 100p_1^{-1}p_2^{-1/2}$

$D_2 = 500\,p_2^{-1}p_1^{-3/2}$

$$\frac{\partial D_1}{\partial p_2} = -50\,p_1^{-1}p_2^{-3/2} < 0$$

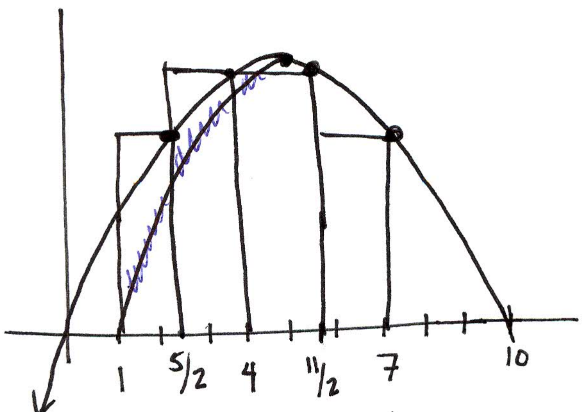
$$\frac{\partial D_2}{\partial p_1} = -750\,p_2^{-1}p_1^{-5/2} < 0$$

The products are complementary.

Examples: hot dogs & hot dog buns, Corona & Lime, ...

5. Using four rectangles, ***estimate*** the area under the curve $y = 10x - x^2$ between $x = 1$ and $x = 7$. Then find the ***exact*** area.

$y = -(x^2 - 10x) = -x(x-10)$ parabola, opens down, (0,0) & (10,0)



total width = 6.

Divide into 4 pieces.

Each has width $\frac{6}{4} = \frac{3}{2}$

Estimate

$$A \approx R_1 + R_2 + R_3 + R_4$$

$$\approx \frac{3}{2}\left(f\left(\tfrac{5}{2}\right)\right) + \frac{3}{2} f(4) + \frac{3}{2} f\left(\tfrac{11}{2}\right) + \frac{3}{2} f(7)$$

$$\approx \frac{3}{2}\left(25 - \frac{25}{4}\right) + \frac{3}{2}(40 - 16)$$

$$+ \frac{3}{2}\left(55 - \frac{121}{4}\right) + \frac{3}{4}(70 - 49)$$

$$\approx \frac{3}{2}\left[\frac{75}{4} + 24 + \frac{99}{4} + 21\right]$$

$$\approx \frac{3}{2}\left(\frac{87}{2} + 45\right) \approx \frac{3}{2}\left(\frac{177}{2}\right) \approx \frac{531}{4}$$

Exact

$$A = \int_1^7 (10x - x^2)\,dx = 5x^2 - \frac{1}{3}x^3 \Big|_1^7$$

$$= \left(5(49) - \frac{343}{3}\right) - \left(5 - \frac{1}{3}\right)$$

$$= 240 - \frac{342}{3} = 240 - 114 = 126$$

6. Calculate $\int_1^{\infty} \frac{x^2}{(x^3+2)^2}\,dx$.

$$= \lim_{n\to\infty} \int_1^n \frac{x^2}{(x^3+2)^2}\,dx = \lim_{n\to\infty} \int_{x=1}^{x=n} \frac{1}{3} u^{-2}\,du = \lim_{n\to\infty} \left[\frac{-1}{3} u^{-1}\right]_{x=1}^{x=n}$$

let $u = x^3 + 2$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$$= \lim_{n\to\infty} \frac{-1}{3(x^3+2)}\Big|_1^n$$

$$= \lim_{n\to\infty} \left[\frac{-1}{3(n^3+2)} + \frac{1}{3(1+2)}\right]$$

$$= \frac{1}{9}$$

7. A computer company has a monthly advertising budget of \$60,000. Its marketing department estimates that if x dollars are spent each month on advertising in electronic media and y dollars per month are spent on television advertising, then the monthly sales will be $S = 90x^{\frac{1}{4}}y^{\frac{3}{4}}$ dollars. If profit is 10% of sales, less the advertising cost, determine how to allocate the advertising budget in order to maximize monthly profit.

$x+y = 60000 \leftarrow$ constraint

Profit $= 0.10$ (sales) $-$ adv. cost

$P = 0.10(90x^{1/4}y^{3/4}) - 60000$

$P = 9x^{1/4}y^{3/4} - 60000 \longleftarrow$ optimize this.

$F(x,y,\lambda) = 9x^{1/4}y^{3/4} - 60000 - \lambda(x+y-60000)$

$= 9x^{1/4}y^{3/4} - 60000 - \lambda x - \lambda y + 60000\lambda$

$F_x = \frac{9}{4}x^{-3/4}y^{3/4} - \lambda = 0 \longrightarrow \lambda = \frac{9y^{3/4}}{4x^{3/4}} = \frac{27x^{1/4}}{4y^{1/4}}$

$F_y = \frac{27}{4}x^{1/4}y^{-1/4} - \lambda = 0$

$9y = 27x$

$y = 3x$

$F_\lambda = -x-y+60000 = 0$

$-x-3x+60000 = 0$

$60000 = 4x$

$\begin{cases} x = 15000 & \text{on electronic media} \\ y = 45000 & \text{on TV} \end{cases}$