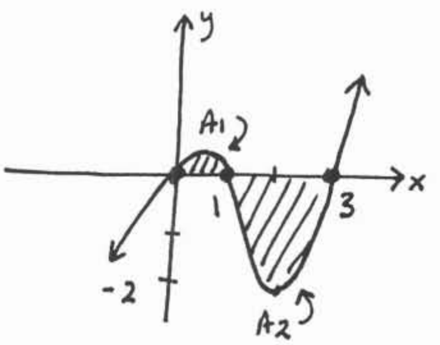
NAME Key

Math 12
Test 4
Spring 2010

You have 50 minutes to complete this test. You must *show all work* to receive full credit. Work any 6 of the following 7 problems. Clearly **CROSS OUT** the problem you do not wish me to grade. Each problem is worth 16 points, and you get 4 points for free, for a total of 100 points. The answers will be posted on the electronic reserves later today.

1. Find the area of the region bounded by $y = x^3 - 4x^2 + 3x$ and the x-axis. Be sure to sketch a graph first!

$y = x(x^2 - 4x + 3)$
$y = x(x-3)(x-1)$
points on graph:
$(0,0), (3,0), (1,0),$
$(2,-2), (1/2, 5/8)$



$$\text{Area} = A_1 + A_2$$
$$= \int_0^1 [(x^3 - 4x^2 + 3x) - 0]\,dx + \int_1^3 [0 - (x^3 - 4x^2 + 3x)]\,dx$$
$$= \left[\tfrac{1}{4}x^4 - \tfrac{4}{3}x^3 + \tfrac{3}{2}x^2\right]_0^1 + \left[\tfrac{-1}{4}x^4 + \tfrac{4}{3}x^3 - \tfrac{3}{2}x^2\right]_1^3$$
$$= \left[\left(\tfrac{1}{4} - \tfrac{4}{3} + \tfrac{3}{2}\right) - 0\right] + \left[\left(\tfrac{-81}{4} + 36 - \tfrac{27}{2}\right) - \left(\tfrac{-1}{4} + \tfrac{4}{3} - \tfrac{3}{2}\right)\right]$$
$$= \frac{5}{12} + \left[\frac{9}{4} - \left(\frac{-5}{12}\right)\right]$$
$$= \frac{5}{12} + \frac{32}{12}$$
$$= \frac{37}{12}$$

2. For $f(x, y) = \dfrac{\ln(x^2 + 5)}{y}$, find all four second-order partial derivatives. Do NOT simplify.

$$f_x = \frac{\left(\frac{1}{x^2+5}\right)(2x)(y) - [\ln(x^2+5)](0)}{y^2} = \frac{2x}{(x^2+5)y}$$

$$f_y = \frac{(0)(y) - [\ln(x^2+5)](1)}{y^2} = \frac{-\ln(x^2+5)}{y}$$

$$f_{xx} = \frac{(2)(x^2+5)(y) - (2x)(2xy)}{[(x^2+5)y]^2} \qquad f_{xy} = \frac{0 - (x^2+5)(2x)}{[(x^2+5)(y)]^2}$$

$$f_{yy} = \frac{0 + [\ln(x^2+5)](2y)}{(y^2)^2} \qquad f_{yx} = \frac{\left(\frac{-1}{x^2+5}\right)(2x)(y^2) - 0}{y^4}$$

equal

3. Find and classify the critical points of $f(x,y) = x^3 - 3xy + y^2 + y - 5$.

$f_x = 3x^2 - 3y = 0 \longrightarrow 3y = 3x^2$

$y = x^2$

$f_y = -3x + 2y + 1 = 0 \longrightarrow -3x + 2x^2 + 1 = 0$

$2x^2 - 3x + 1 = 0$

$(x-1)(2x-1) = 0$

crit pts: $(1,1)$, $(\frac{1}{2}, \frac{1}{4})$

$f_{xx} = 6x$

$f_{yy} = 2$

$f_{xy} = -3$

$D(x,y) = 12x - 9$

$D(1,1) = 12 - 9 > 0$, $f_{xx}(1,1) = 6 > 0$, so $(1,1)$ gives a minimum

$D(\frac{1}{2}, \frac{1}{4}) = 6 - 9 < 0$, so $(\frac{1}{2}, \frac{1}{4})$ gives a saddle point

4. Suppose p_1 and p_2 are the prices of two products. Also suppose $D_1(p_1, p_2) = \dfrac{100}{p_1\sqrt{p_2}}$ and $D_2(p_1, p_2) = \dfrac{500}{p_2\sqrt[3]{p_1}}$ are the demand functions for the two products (quantities). Answer the following questions, showing your work below.

a) Are these two products competitive (substitutes), complementary, or neither? (show work)

$D_1 = 100 p_1^{-1} p_2^{-1/2}$ $\quad D_2 = 500 p_2^{-1} p_1^{-1/3}$

$\frac{\partial D_1}{\partial p_2} = -50 p_1^{-1} p_2^{-3/2} < 0$

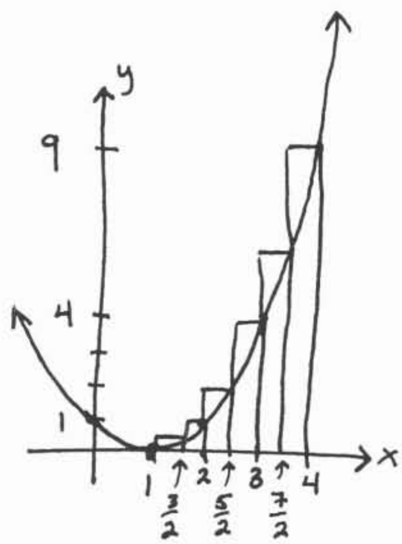
$\frac{\partial D_2}{\partial p_1} = \frac{-500}{3} p_2^{-1} p_1^{-4/3} < 0$

both are negative, so products are <u>complementary</u>

b) An example of two products that might behave this way are

hot dogs and hot dog buns.

5. Using six rectangles, ***estimate*** the area under the curve $y = (x-1)^2$ between $x = 1$ and $x = 4$. Then find the ***exact*** area.



widths are all = 1/2

$f(1) = 0$
$f(3/2) = 1/4$
$f(2) = 1$
$f(5/2) = 9/4$
$f(3) = 4$
$f(7/2) = 25/4$
$f(4) = 9$

} heights of rectangles

$$A \approx R_1 + R_2 + R_3 + R_4 + R_5 + R_6$$

$$\approx \left(\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right) + (1)\left(\tfrac{1}{2}\right) + \left(\tfrac{9}{4}\right)\left(\tfrac{1}{2}\right) + (4)\left(\tfrac{1}{2}\right) + \left(\tfrac{25}{4}\right)\left(\tfrac{1}{2}\right) + (9)\left(\tfrac{1}{2}\right)$$

$$\approx \frac{1}{8} + \frac{4}{8} + \frac{9}{8} + \frac{16}{8} + \frac{25}{8} + \frac{36}{8} \approx \frac{91}{8}$$

Exact: $A = \int_1^4 (x-1)^2 dx$ $u = x-1$, $du = dx$

$$= \int_1^4 u^2 du$$

$$= \tfrac{1}{3} u^3 \Big|_1^4 = \tfrac{1}{3}(x-1)^3 \Big|_1^4 = \tfrac{1}{3}(3)^3 - \tfrac{1}{3}(0)^3 = 9$$

6. Calculate $\int_0^\infty x e^{-x^2} dx$.

$$= \lim_{n \to \infty} \int_0^n x e^{-x^2} dx$$

$u = -x^2$
$du = -2x\, dx$
$\frac{-1}{2} du = x\, dx$

$$= \lim_{n \to \infty} \frac{-1}{2} \int_0^n e^u du$$

$$= \lim_{n \to \infty} \frac{-1}{2} \left[e^{-x^2} \right]_0^n$$

$$= \lim_{n \to \infty} \frac{-1}{2} \left(\underbrace{e^{-n^2}}_{\to 0} - e^0 \right)$$

$$= \frac{-1}{2}(0 - 1)$$

$$= \frac{1}{2}$$

7. Suppose that a firm produces x units of one part and y units of another needed to make a final product, and the total number of finished items produced is given by $P = 12x + 20y - x^2 - 2y^2$. It costs \$4 to produce a unit of the first part, and \$8 to produce a unit of the second part. Under a budget constraint of \$88, what is the largest number of *final products* that can be produced?

maximize $P = 12x + 20y - x^2 - 2y^2$ subject to $4x + 8y = 88$

$F(x, y, \lambda) = 12x + 20y - x^2 - 2y^2 - \lambda(4x + 8y - 88)$.

$F_x = 12 - 2x - 4\lambda = 0 \longrightarrow 6 - x - 2\lambda = 0 \rightarrow x = 6 - 2\lambda$

$F_y = 20 - 4y - 8\lambda = 0 \longrightarrow 5 - y - 2\lambda = 0 \rightarrow y = 5 - 2\lambda$

$F_\lambda = -4x - 8y + 88 = 0 \longrightarrow -4(6 - 2\lambda) - 8(5 - 2\lambda) + 88 = 0$

$-24 + 8\lambda - 40 + 16\lambda + 88 = 0$

$24\lambda + 24 = 0$

$\lambda = -1$

$x = 6 + 2 = 8$

$y = 5 + 2 = 7$

8 units of first part
7 units of second part

max # of final products is

$P(8, 7) = 12(8) + 20(7) - (8)^2 - 2(7)^2$

$= 96 + 140 - 64 - 98$

$= 236 - 162$

$= 74$

74 finished products is the max