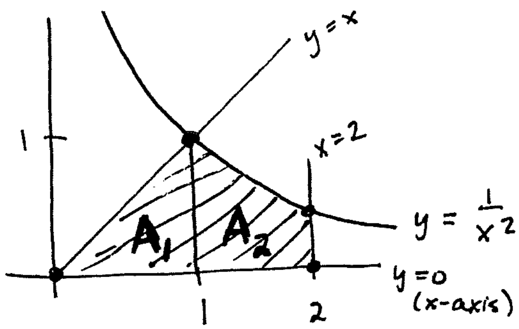
NAME KEY

Math 12
Test 4
Spring 2013

You have 50 minutes to complete this test. You must *show all work* to receive full credit. Work any 6 of the following 7 problems. Clearly **CROSS OUT** the problem you do not wish me to grade. Each problem is worth 16 points, and you get 4 points for free, for a total of 100 points. The answers will be posted on the electronic reserves later today.

1. Find the area of the region bounded by the curves $y = x$, $y = \frac{1}{x^2}$, $y = 0$, and $x = 2$. Be sure to sketch a graph first!



$$\text{Area} = A_1 + A_2$$
$$= \int_0^1 x\,dx + \int_1^2 \frac{1}{x^2}\,dx$$
$$= \left[\frac{1}{2}x^2\right]_0^1 + \left[-x^{-1}\right]_1^2$$
$$= \left[\frac{1}{2}(1) - \frac{1}{2}(0)\right] + \left[-\frac{1}{2} - \left(\frac{-1}{1}\right)\right]$$
$$= \frac{1}{2} + \left(-\frac{1}{2} + 1\right)$$
$$= 1$$

2. Suppose $z = xe^{x-y} - ye^{y-x}$. Show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = e^{x-y} - e^{y-x}$.

$$\frac{\partial z}{\partial x} = \left[(1)(e^{x-y}) + (x)(e^{x-y})(1)\right] - \left[(0)(e^{y-x}) + (y)(e^{y-x})(-1)\right]$$
$$= e^{x-y} + xe^{x-y} + ye^{y-x}$$
$$\frac{\partial z}{\partial y} = \left[(0)(e^{x-y}) + (x)(e^{x-y})(-1)\right] - \left[(1)(e^{y-x}) + (y)(e^{y-x})(1)\right]$$
$$= -xe^{x-y} - e^{y-x} - ye^{y-x}$$
$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = e^{x-y} \cancel{+ xe^{x-y}} \cancel{+ ye^{y-x}} \cancel{- xe^{x-y}} - e^{y-x} \cancel{- ye^{y-x}}$$
$$= e^{x-y} - e^{y-x} \checkmark$$

3. Find and classify the critical points of $f(x,y) = x^3 - 3xy + y^2 + y - 5$.

$f_x = 3x^2 - 3y = 0 \longrightarrow x^2 - y = 0 \rightarrow y = x^2$

$f_y = -3x + 2y + 1 = 0 \longrightarrow -3x + 2x^2 + 1 = 0$

$2x^2 - 3x + 1 = 0$

$(2x-1)(x-1) = 0$

$x = 1/2 \quad x = 1$

$y = \frac{1}{4} \quad y = 1$

Critical Points: $(\frac{1}{2}, \frac{1}{4})$ and $(1,1)$

$$\left.\begin{aligned} f_{xx} &= 6x \\ f_{yy} &= 2 \\ f_{xy} &= -3 \end{aligned}\right\} \quad D(x,y) = (6x)(2) - (-3)^2 = 12x - 9$$

$D(\frac{1}{2}, \frac{1}{4}) = 6 - 9 < 0$, so $(\frac{1}{2}, \frac{1}{4})$ gives a saddle point.

$D(1,1) = 12 - 9 > 0$, and $f_{xx}(1,1) = 6 > 0$

so $(1,1)$ gives a minimum.

up, min

4. The demand functions for two products are given by $D_1 = \dfrac{100}{p_1\sqrt{p_2}}$ and $D_2 = \dfrac{500}{p_2\sqrt[3]{p_1}}$, where p_1 and p_2 are the respective prices of the products. Are the two products competitive, complementary, or neither? (show work, and remember that prices are positive numbers) Give an example of two products that might behave in this way.

$D_1 = 100\, p_1^{-1} p_2^{-1/2}$

$\dfrac{\partial D_1}{\partial p_2} = -50\, p_1^{-1} p_2^{-3/2}$

< 0 since $p_1, p_2 \geq 0$

$D_2 = 500\, p_2^{-1} p_1^{-3/2}$

$\dfrac{\partial D_2}{\partial p_1} = -750\, p_2^{-1} p_1^{-5/2}$

< 0 since $p_1, p_2 \geq 0$

The two products are complementary. An example could be hot dogs and hot dog buns, or Corona & limes, etc...

5. When x units of labor and y units of capital are invested, a manufacturer's total production is given by the function $Q = 5x^{\frac{1}{5}}y^{\frac{4}{5}}$ units. Each unit of labor costs \$22 and each unit of capital costs \$66. If exactly \$23,760 is to be spent on production, determine the number of units of labor and the number of units of capital that should be invested to maximize production.

maximize $Q = 5x^{1/5}y^{4/5}$ subject to $22x + 66y = 23760$

$F(x, y, \lambda) = 5x^{1/5}y^{4/5} - \lambda(22x + 66y - 23760)$

$F_x = x^{-4/5}y^{4/5} - 22\lambda = 0 \longrightarrow \lambda = \dfrac{y^{4/5}}{22x^{4/5}}$

$F_y = 4x^{1/5}y^{-1/5} - 66\lambda = 0 \longrightarrow \lambda = \dfrac{4x^{1/5}}{66y^{1/5}}$

$\dfrac{y^{4/5}}{22x^{4/5}} = \dfrac{2x^{1/5}}{33y^{1/5}}$

$F_\lambda = -22x - 66y + 23760 = 0$

$33y = 44x$

$y = \frac{4}{3}x$

$-22x - \overset{22}{\cancel{66}}\left(\frac{4}{\cancel{3}}x\right) + 23760 = 0$

$23760 = 22x + 88x = 110x$

$216 = x, \quad y = \frac{4}{3}(216) = 288$

To maximize production, invest 216 units of labor and 288 units of capital.

6. Solve $\int_0^\infty 5e^{-x}\,dx$.

$$\int_0^\infty 5e^{-x}\,dx = \lim_{n\to\infty}\int_0^n 5e^{-x}\,dx$$

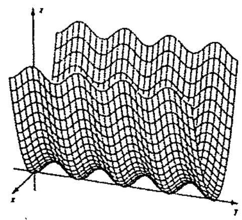
$$= \lim_{n\to\infty}\left[-5e^{-x}\right]_0^n$$

$$= \lim_{n\to\infty}\left[-5e^{-n} - (-5e^{0})\right]$$

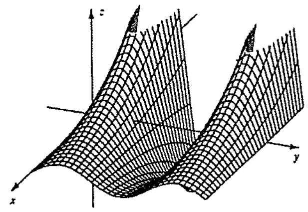
$$= \lim_{n\to\infty}\left(\underbrace{\frac{-5}{e^n}}_{0} + 5\right)$$

$$= 5$$

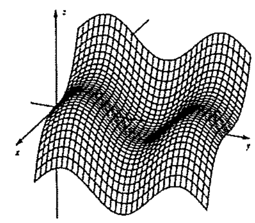
7. Match the surfaces (top 6 graphs) with the lettered level curves (bottom 6 graphs).



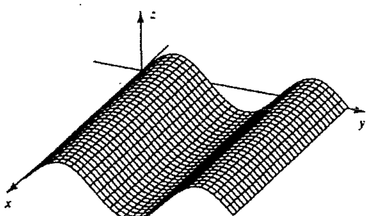
Matches B



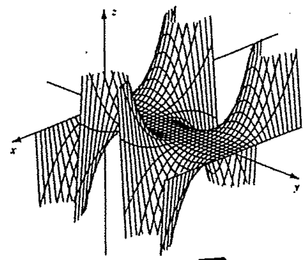
Matches E



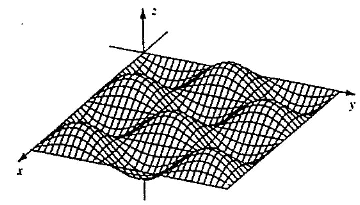
Matches D



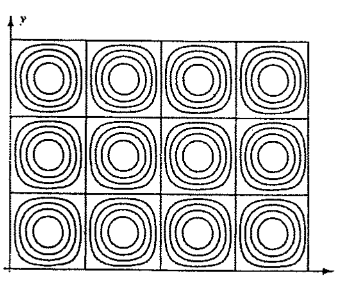
Matches C



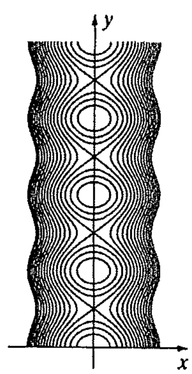
Matches F



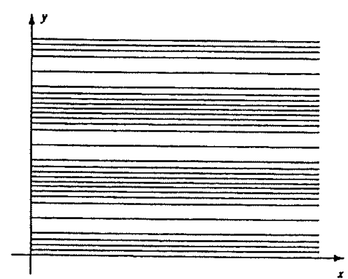
Matches A



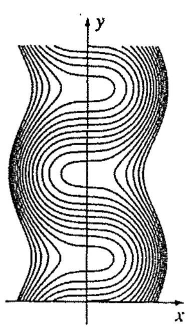
A



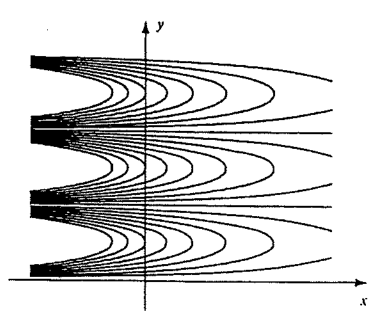
B



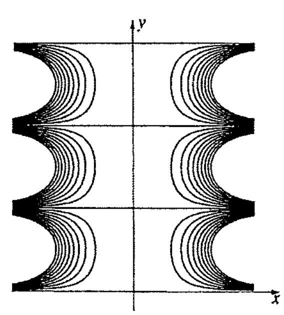
C



D



E



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