Mathematics 204

Fall 2012

Exam I

[1] Your Printed Name: Dr. Grow

[1] Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic noisemaking devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam I consists of this cover page and 6 pages of problems containing 6 numbered problems.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [17] at the beginning of a problem indicates the point value of that problem is 17. The maximum possible score on this exam is 100.

	0	1	2	3	4	5	6	**Sum**
points earned								
maximum points	2	15	17	17	17	18	14	100

1.[15] Determine all values of r for which the differential equation $4t^2y''+12ty'+3y=0$ has solutions of the form $y=t^r$ for $t>0$.

Let $y=t^r$ be a solution of the DE. Then $y'=rt^{r-1}$ and $y''=r(r-1)t^{r-2}$ so

$$4t^2y''+12ty'+3y=0$$

implies, upon substitution of the expressions for y, y', and y'', that

$$4t^2r(r-1)t^{r-2}+12trt^{r-1}+3t^r=0.$$

Since $t^2\cdot t^{r-2}=t^r$ and $t\cdot t^{r-1}=t^r$, this means

$$4r(r-1)t^r+12rt^r+3t^r=0$$

or

$$(4r(r-1)+12r+3)t^r=0.$$

But $t>0$ so $t^r\neq 0$ and hence

$$4r(r-1)+12r+3=0.$$

Rearranging terms and factoring gives

$$4r^2+8r+3=0$$

$$(2r+3)(2r+1)=0.$$

Therefore $\boxed{r=-3/2 \quad \text{or} \quad r=-1/2}$. That is, $y_1=t^{-3/2}$ and $y_2=t^{-1/2}$

solve the DE on the interval $0<t<\infty$.

2.[17] Solve the initial value problem $4t^2 - ty' = 2y,\ y(1) = 2.$

The DE is a first order linear equation: $ty' + 2y = 4t^2$. We normalize and compute an integrating factor: $y' + \frac{2}{t}y = 4t$ so $\mu(t) = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln(t) + \cancel{c}^{\,0}} = e^{\ln(t^2)} = t^2$. Multiplying the normalized DE by the integrating factor yields

$$t^2\left(y' + \frac{2}{t}y\right) = t^2(4t)$$

or

$$t^2y' + 2ty = 4t^3.$$

But the left member of the last equation is exact: $\frac{d}{dt}\left[t^2y\right] = t^2y' + 2ty$.
Therefore the last equation can be rewritten as

$$\frac{d}{dt}\left[t^2y\right] = 4t^3.$$

Integrating both sides with respect to t yields

$$t^2y = \int 4t^3dt = t^4 + c,$$

so the general solution of the DE is

$$y(t) = \frac{t^4+c}{t^2} = t^2 + \frac{c}{t^2}.$$

We apply the initial condition to determine the constant c:

$$2 = y(1) = 1^2 + \frac{c}{1^2} = 1 + c \quad \Rightarrow \quad c = 1.$$

Therefore $\boxed{y(t) = t^2 + \frac{1}{t^2}}$ solves the I.V.P.

3.[17] Solve the initial value problem $e^t y' - \left(\frac{1+e^t}{y+1}\right) y = 0,\ y(0) = 1.$

The D.E. is first order, nonlinear, and separable:

$$e^t \frac{dy}{dt} = (1+e^t)\left(\frac{y}{y+1}\right) \iff \frac{y+1}{y}\, dy = \frac{1+e^t}{e^t}\, dt.$$

Simplifying and integrating both sides yields

$$y + \ln|y| = \int \left(1 + \frac{1}{y}\right) dy = \int \left(\frac{1}{e^t} + 1\right) dt = \int (e^{-t} + 1)\, dt = -e^{-t} + t + c$$

Applying the initial condition gives

$$1 + \ln(1) = -e^{-0} + 0 + c \implies c = 2.$$

Therefore $\boxed{y + \ln(y) + e^{-t} - t = 2}$ is an (implicit) solution to the I.V.P. (Note that we have dropped the absolute value on y inside the logarithm because $y(0) = 1 > 0$ and so $y > 0$ on the maximal interval of existence of the solution.)

4.[17] A 1000 gallon tank originally holds 300 gallons of water solution containing 100 pounds of salt. Then water containing 2 pounds of salt per gallon is poured into the tank at a rate of 5 gallons per minute, and the well-stirred mixture is allowed to leave the tank at a rate of 3 gallons per minute.
(a) How long will it take before the tank begins to overflow?
(b) Set up, **BUT DO NOT SOLVE**, an initial value problem that models the amount of salt in the tank at all times prior to the moment when the tank overflows.

(a) The volume $V(t)$ of solution in the tank at time t minutes satisfies $\frac{dV}{dt} = 5-3$ gallons per minute. Therefore $V(t) = 2t + 300$. The tank overflows when $V(t) = 1000$ gallons. Solving $2t+300 = 1000$ gives $t = 350$ minutes (or 5 hours and 50 minutes).

(b) Let $A(t)$ be the number of pounds of salt in the tank at time t minutes where $0 \le t \le 350$. We use

Net Rate of Change = Inflow Rate − Outflow Rate.

Therefore
$$\frac{dA}{dt} = \left(\frac{2 \text{ lbs.}}{\text{gal}}\right)\left(\frac{5 \text{ gal}}{\text{min}}\right) - \left(\frac{A(t) \text{ lbs.}}{V(t) \text{ gal.}}\right)\left(\frac{3 \text{ gal}}{\text{min}}\right).$$

From part (a), $V(t) = 2t+300$, so substituting in the previous equation and simplifying yields

$$\boxed{\begin{aligned} \frac{dA}{dt} &= 10 - \left(\frac{3}{2t+300}\right)A \\ A(0) &= 100 . \end{aligned}}$$

where

(Here A is in pounds and t is in minutes.)

5.[18] Find the general solutions to the following differential equations.

(a) $y''-3y'=0$ $\quad y=e^{rt}$ leads to $r^2-3r=0 \Leftrightarrow r(r-3)=0$. Thus $r=0$ or $r=3$. Consequently, $\boxed{y=c_1+c_2e^{3t}}$ is the general solution of the D.E. where c_1 and c_2 are arbitrary constants.

[$W(1,e^{3t}) = \begin{vmatrix} 1 & e^{3t} \\ 0 & 3e^{3t} \end{vmatrix} = 3e^{3t} \neq 0$ so $\{1, e^{3t}\}$ is a fundamental set of solutions to $y''-3y'=0$.]

(b) $y''-2y'+5y=0$ $\quad y=e^{rt}$ leads to $r^2-2r+5=0$ so $r=\frac{2\pm\sqrt{4-20}}{2}=\frac{2\pm 4i}{2}$ $=1\pm 2i$. Consequently, $\boxed{y=c_1e^{t}\cos(2t)+c_2e^{t}\sin(2t)}$ is the general solution of the D.E. where c_1 and c_2 are arbitrary constants.

[$W(e^{t}\cos 2t, e^{t}\sin 2t) = \begin{vmatrix} e^{t}\cos(2t) & e^{t}\sin(2t) \\ e^{t}\cos(2t)-2e^{t}\sin(2t) & e^{t}\sin(2t)+2e^{t}\cos(2t) \end{vmatrix} = \begin{array}{l} e^{2t}\cos(2t)\sin(2t)+2e^{2t}\cos^2(2t) \\ -e^{2t}\cos(2t)\sin(2t)+2e^{2t}\sin^2(2t) \end{array}$

$=2e^{2t}\neq 0$ so $\{e^{t}\cos(2t), e^{t}\sin(2t)\}$ is a F.S.S. to $y''-2y'+5y=0$.]

(c) $y''+6y'+9y=0$ $\quad y=e^{rt}$ leads to $r^2+6r+9=0$ so $(r+3)^2=0$ and $r=-3$ (multiplicity two). Consequently, $\boxed{y=c_1e^{-3t}+c_2te^{-3t}}$ is the general solution of the D.E. where c_1 and c_2 are arbitrary constants.

[$W(e^{-3t}, te^{-3t}) = \begin{vmatrix} e^{-3t} & te^{-3t} \\ -3e^{-3t} & e^{-3t}-3te^{-3t} \end{vmatrix} = e^{-6t}-3te^{-6t}+3te^{-6t} = e^{-6t}\neq 0$ so

$\{e^{-3t}, te^{-3t}\}$ is a F.S.S. to $y''+6y'+9y=0$.]

6.[14] Define the function f by the formula

$$f(t,y)=\begin{cases}(t-y)^2 & \text{if } t\le y,\\ (t-y)^{3/2} & \text{if } t>y.\end{cases}$$

For what values of t_0 and y_0 does the initial value problem

$$y'=f(t,y),\ \ y(t_0)=y_0,$$

have a unique solution defined on some open interval $t_0-h<t<t_0+h$ containing t_0?

If $t<y$ then $f(t,y)=(t-y)^2$ and $\frac{\partial f}{\partial y}=-2(t-y)$ are continuous at (t,y).

If $t>y$ then $f(t,y)=(t-y)^{3/2}$ and $\frac{\partial f}{\partial y}=-\frac{3}{2}(t-y)^{1/2}$ are continuous at (t,y).

Suppose $t=y=k$. Then

$$\lim_{\substack{(T,Y)\to(k,k)\\ T\le Y}} f(T,Y)=\lim_{\substack{(T,Y)\to(k,k)\\ T\le Y}}(T-Y)^2=0=f(k,k)$$

and

$$\lim_{\substack{(T,Y)\to(k,k)\\ T>Y}} f(T,Y)=\lim_{\substack{(T,Y)\to(k,k)\\ T>Y}}(T-Y)^{3/2}=0=f(k,k).$$

Therefore $\lim_{(T,Y)\to(k,k)} f(T,Y)=f(k,k)$ so f is continuous at (t,y) if $t=y$.

Observe that $$\frac{\partial f}{\partial y}(t,y)=\begin{cases}-2(t-y) & \text{if } t<y,\\ 0 & \text{if } t=y,\\ -\frac{3}{2}(t-y)^{1/2} & \text{if } t>y,\end{cases}$$

so $\frac{\partial f}{\partial y}$ is continuous at (t,y) if $t=y$ by the same type of reasoning.

Since f and $\frac{\partial f}{\partial y}$ are continuous in a neighborhood of (t_0,y_0) for all real t_0 and y_0, the existence and uniqueness theorem guarantees $\boxed{\text{the IVP } y'=f(t,y),\ y(t_0)=y_0, \text{ has a unique solution for every real number } t_0 \text{ and every real number } y_0.}$

2012 Fall Semester, Math 204 Hour Exam I
Instructor Grow, Section M

100			59	||		19
99			58	||||		18
98			57			17
97			56			16
96			55	|		15
95	|		54			14
94	|	3 As	53	|	13 Fs	13
93			52			12
92			51			11
91			50			10
90	|		49			9
89			48			8
88			47	|		7
87	|		46			6
86	|		45			5
85	|		44	|		4
84		8 Bs	43			3
83			42	|		2
82			41			1
81	|||		40			0
80	||		39			
79			38			
78			37			
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64		3 Ds	23			
63	|		22			
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61	|		20			
60	|					

Number taking exam: 39
Median: 74
Mean: 68.4
Standard Deviation: 16.4

Number receiving A's: 3
Number receiving B's: 8
Number receiving C's: 12
Number receiving D's: 3
Number receiving F's: 13