Mathematics 204

Fall 2013

Exam II

Your Printed Name: Dr. Grow

Your Instructor's Name: ______________________

Your Section (or Class Meeting Days and Time): ______________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam II consists of this cover page and 7 pages of problems containing 7 numbered problems.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [16] at the beginning of a problem indicates the point value of that problem is 16. The maximum possible score on this exam is 100.

problem	1	2	3	4	5	6	7	**Sum**
points earned								
maximum points	16	16	16	12	12	16	12	100

1.[16] Find the general solution of each differential equation.

(a) $y^{(4)} - 2y''' + 2y'' - 2y' + y = 0$

$y = e^{rt}$ leads to $r^4 - 2r^3 + 2r^2 - 2r + 1 = 0$. By inspection $r = 1$ is a solution. We divide the polynomial in the left member by $r-1$ synthetically:

$$\begin{array}{r|rrrrr}
1 & 1 & -2 & 2 & -2 & 1 \\
 & & 1 & -1 & 1 & -1 \\
\hline
1 & 1 & -1 & 1 & -1 & 0 = R \\
 & & 1 & 0 & 1 & \\
\hline
 & 1 & 0 & 1 & 0 = R &
\end{array}$$

Therefore the characteristic equation factors as $(r-1)(r-1)(r^2+1) = 0$. The roots are $r = 1$ (multiplicity two) and $r = \pm i$. The general solution of the DE is

$$\boxed{y(t) = c_1 e^t + c_2 t e^t + c_3 \cos(t) + c_4 \sin(t)}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.

(b) $t^2 y'' + ty' + y = 0$

$y = t^m$ leads to $m(m-1) + m + 1 = 0$ or $m^2 + 1 = 0$. The roots are $m = \pm i$. Therefore $y_1 = t^i$ and $y_2 = t^{-i}$ are two (complex-valued) solutions. To get real-valued solutions, we write

$$y_1 = t^i = (e^{\ln t})^i = e^{i\ln t} = \cos(\ln t) + i\sin(\ln t)$$
$$y_2 = t^{-i} = (e^{\ln t})^{-i} = e^{-i\ln t} = \cos(\ln t) - i\sin(\ln t).$$

[Here we have used the Euler identity $e^{i\theta} = \cos\theta + i\sin\theta$.] Then

$$\tilde{y}_1 = \frac{y_1 + y_2}{2} = \cos(\ln t)$$

$$\tilde{y}_2 = \frac{y_1 - y_2}{2i} = \sin(\ln t)$$

is a fundamental set of real-valued solutions of the DE on the interval $t > 0$. Thus

$$\boxed{y(t) = c_1 \cos(\ln t) + c_2 \sin(\ln t)} \quad (c_1, c_2 \text{ arbitrary constants})$$

is the general solution of the DE on $t > 0$.

2.[16] Find a particular solution of $y^{(4)}+3y'''+3y''+y'=t^2$.

$y_h=e^{rt}$ in the associated homogeneous DE, $y^{(4)}+3y'''+3y''+y'=0$, leads to $r^4+3r^3+3r^2+r=0$. Thus $r(r^3+3r^2+3r+1)=0$ or $r(r+1)^3=0$ so the roots are $r=0$ and $r=-1$ (multiplicity 3). Consequently $y_h=c_1+c_2e^{-t}+c_3te^{-t}+c_4t^2e^{-t}$.

Since $g(t)=t^2$, a trial particular solution is $y_p=t^s(At^2+Bt+C)$ where A, B, and C are constants to be determined and s is the smallest nonnegative integer such that no term in y_p is a solution to the associated homogeneous equation. Note that the term c_1 appears in y_h so we must take $s=1$. Then

$$y_p=t(At^2+Bt+C)=At^3+Bt^2+Ct$$
$$y_p'=3At^2+2Bt+C$$
$$y_p''=6At+2B$$
$$y_p'''=6A$$
$$y_p^{(4)}=0.$$

We want $y_p^{(4)}+3y_p'''+3y_p''+y_p'=t^2$, so substituting from above gives

$$3(6A)+3(6At+2B)+3At^2+2Bt+C=t^2.$$

Rearranging and equating like coefficients,

$$3At^2+(18A+2B)t+(18A+6B+C)=1\cdot t^2+0\cdot t+0\cdot t^0$$

so $3A=1$, $18A+2B=0$, and $18A+6B+C=0$. Thus $A=1/3$, $B=-9A=-3$, and $C=-18A-6B=-6+18=12$. Consequently

$$\boxed{y_p(t)=\frac{1}{3}t^3-3t^2+12t}$$

is a particular solution of the DE $y^{(4)}+3y'''+3y''+y'=t^2$.

3.[16] Find the general solution of $y'' - 2y' + y = t^{-1}e^{t}$ on the interval $t > 0$.

$y_h = e^{rt}$ in the associated homogeneous equation $y'' - 2y' + y = 0$ leads to $r^2 - 2r + 1 = 0$ or $(r-1)(r-1) = 0$ so $r = 1$ (multiplicity two). Hence $y_1(t) = e^t$ and $y_2(t) = te^t$ is a fundamental set of solutions to the homogeneous equation since

$$W(y_1,y_2)(t) = \begin{vmatrix} e^t & te^t \\ e^t & te^t + e^t \end{vmatrix} = te^{2t} + e^{2t} - te^{2t} = e^{2t} \neq 0.$$

A particular solution on the nonhomogeneous equation is $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ where

$$u_1(t) = \int \frac{-g(t)y_2(t)}{W(y_1,y_2)(t)}\,dt = \int \frac{-t^{-1}e^t \cdot te^t}{e^{2t}}\,dt = \int -1\,dt = -t + \cancel{c_1}^{\,0}$$

and

$$u_2(t) = \int \frac{g(t)y_1(t)}{W(y_1,y_2)(t)}\,dt = \int \frac{t^{-1}e^t \cdot e^t}{e^{2t}}\,dt = \int t^{-1}\,dt = \ln(t) + \cancel{c_2}^{\,0}.$$

Therefore $y_p(t) = -te^t + \ln(t)\cdot te^t$. The general solution of the nonhomogeneous equation is

$$y = y_h + y_p = c_1e^t + c_2te^t - te^t + t\ln(t)e^t$$

$$\boxed{y(t) = c_1e^t + \tilde{c}_2te^t + t\ln(t)e^t} \qquad (t > 0)$$

where c_1 and $\tilde{c}_2$ are arbitrary constants.

4.[12] A spring hangs vertically from a rigid support. When a 2 pound box is attached to the end of the spring, the box stretches the spring 0.5 feet and then comes to rest. Suppose the box is lifted up 2 feet from the rest position and then released. Assuming the air resistance (or the damping force) at any instant is equal to half the instantaneous velocity of the box, write **BUT DO NOT SOLVE**, an initial value problem modeling the motion of the box.

Let $u(t)$ denote the downward displacement of the box from its static equilibrium position (i.e. from where the box comes to rest) at time t. Then

$$mu'' + \gamma u' + ku = 0$$

is the equation of motion for the box. Here

$$m = \text{mass of box} = \frac{\text{weight}}{\text{acceleration of gravity}} = \frac{2}{32.2} \text{ slugs}$$ (or $\frac{2}{32}$ if a less accurate value of g is used)

$$ku_0 = \text{weight} \Rightarrow k = \frac{2}{0.5} = 4 \text{ lbs/ft.}$$

$$-\text{damping force} = \gamma u' = \left(\tfrac{1}{2}\right)(\text{instantaneous velocity of box}) \Rightarrow \gamma = \frac{1}{2} \text{ lb.ft/sec.}$$

Also $u(0)$ = initial displacement of box = -2 ft.

and $u'(0)$ = initial velocity of box = 0 ft./sec.

Thus

$$\boxed{\frac{1}{16.1}u'' + \frac{1}{2}u' + 4u = 0, \quad u(0) = -2, \quad u'(0) = 0}$$

is the IVP modeling the motion of the box.

5.[12] Determine whether the set of functions

$$f_1(t) = t^2 + 1,\ \ f_2(t) = 2t^2 + t,\ \ f_3(t) = t - 2,$$

is linearly dependent or linearly independent. If they are linearly dependent, find a linear relation among them.

expand by cofactors of row 3

$$W(f_1, f_2, f_3)(t) = \begin{vmatrix} t^2+1 & 2t^2+t & t-2 \\ 2t & 4t+1 & 1 \\ 2 & 4 & 0 \end{vmatrix} = 2\begin{vmatrix} 2t^2+t & t-2 \\ 4t+1 & 1 \end{vmatrix} - 4\begin{vmatrix} t^2+1 & t-2 \\ 2t & 1 \end{vmatrix} + 0$$

$$= 2\left(2t^2+t - (4t^2-7t-2)\right) - 4\left(t^2+1 - (2t^2-4t)\right)$$

$$= 2(-2t^2+8t+2) - 4(-t^2+4t+1)$$

$$= 0.$$

Therefore $\boxed{f_1, f_2, \text{and } f_3 \text{ are linearly dependent}}$. (They are solutions to $y''' = 0$ and their Wronskian vanishes.)

To find a linear relation among them, suppose $c_1 f_1(t) + c_2 f_2(t) + c_3 f_3(t) = 0$ for all real t and some constants c_1, c_2, and c_3. I.e. suppose

$$c_1(t^2+1) + c_2(2t^2+t) + c_3(t-2) = 0.$$

Then

$$(c_1 + 2c_2)t^2 + (c_2 + c_3)t + (c_1 - 2c_3) = 0.$$

Consequently the constants must satisfy the system:

$$\begin{cases} c_1 + 2c_2 \qquad = 0 \\ \qquad c_2 + c_3 = 0 \\ c_1 \qquad - 2c_3 = 0 \end{cases} \xrightarrow{-1E_1 + E_3} \begin{cases} c_1 + 2c_2 \qquad = 0 \\ \qquad c_2 + c_3 = 0 \\ -2c_2 - 2c_3 = 0 \end{cases} \xrightarrow{2E_2 + E_3} \begin{cases} c_1 + 2c_2 \qquad = 0 \\ \qquad c_2 + c_3 = 0 \\ \qquad\qquad 0 = 0. \end{cases}$$

Therefore $c_1 = -2c_2$ and $c_3 = -c_2$ where we are free to choose c_2. Take $c_2 = 1$; then $c_1 = -2$ and $c_3 = -1$. Thus

$$\boxed{-2(t^2+1) + 1(2t^2+t) - 1(t-2) = 0}$$

is a linear relation among the 3 functions.

6.[16] (a) Let $f = f(t)$ be a function defined on the interval $0 \le t < \infty$. State the definition of $\mathcal{L}\{f\}(s)$, the Laplace transform of f at s.

$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}\,dt$ for those values of s for which the improper integral converges.

(b) Use the definition of the Laplace transform to find the Laplace transform of the function

$$f(t) = \begin{cases} \pi & \text{if } 0 \le t < \pi, \\ t & \text{if } \pi \le t < \infty. \end{cases}$$

For which values of s is the Laplace transform of f defined?

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\pi f(t)e^{-st}\,dt + \int_\pi^\infty f(t)e^{-st}\,dt = \int_0^\pi \pi e^{-st}\,dt + \lim_{M\to\infty}\int_\pi^M \underbrace{t}_{u}\,\underbrace{e^{-st}\,dt}_{dv}$$

$$= \left.\frac{\pi e^{-st}}{-s}\right|_{t=0}^{\pi} + \lim_{M\to\infty}\left[\left.\frac{te^{-st}}{-s}\right|_{t=\pi}^{M} - \int_\pi^M \frac{e^{-st}}{-s}\,dt\right]$$

$$= \frac{\pi e^{-s\pi} - \pi}{-s} + \lim_{M\to\infty}\left[\frac{M}{-se^{sM}} + \frac{\pi e^{-s\pi}}{s} - \left.\frac{e^{-st}}{s^2}\right|_{t=\pi}^{M}\right]$$

$$= \frac{\pi - \pi e^{-s\pi}}{s} + \lim_{M\to\infty}\left[\frac{M}{-se^{sM}} + \frac{\pi e^{-s\pi}}{s} - \frac{e^{-sM}}{s^2} + \frac{e^{-s\pi}}{s^2}\right]$$

If $s > 0$ then $\lim\limits_{M\to\infty}\dfrac{e^{-sM}}{s^2} = 0$ and $\lim\limits_{M\to\infty}\dfrac{M}{-se^{sM}} \overset{\text{l'Hospital}}{=} \lim\limits_{M\to\infty}\dfrac{1}{-s^2e^{sM}} = 0$.

If $s \le 0$ then the improper integral does not converge. Thus, $\boxed{\text{for } s>0}$,

$$\mathcal{L}\{f\}(s) = \frac{\pi}{s} - \cancel{\frac{\pi e^{-s\pi}}{s}} + \cancel{\frac{\pi e^{-s\pi}}{s}} + \frac{e^{-s\pi}}{s^2} = \boxed{\frac{\pi}{s} + \frac{e^{-s\pi}}{s^2}}.$$

7.[12] Find the inverse Laplace transform of $F(s)=\dfrac{s+1}{s^2-s-6}$.

Partial Fraction Decomp.

$$F(s)=\frac{s+1}{(s-3)(s+2)} \overset{\downarrow}{=} \frac{A}{s-3}+\frac{B}{s+2}.$$

Then $$s+1=\left(\frac{A}{s-3}+\frac{B}{s+2}\right)(s-3)(s+2)$$

$$s+1=A(s+2)+B(s-3).$$

To find A, set $s=3$: $\quad 4=5A \Rightarrow A=4/5$.

To find B, set $s=-2$: $\quad -1=-5B \Rightarrow B=1/5$.

Therefore

$$\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{4/5}{s-3}+\frac{1/5}{s+2}\right\}=\boxed{\frac{4}{5}e^{3t}+\frac{1}{5}e^{-2t}}.$$

2013 Fall Semester, Math 204 Hour Exam II
Instructor Grow, Section D

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Number taking exam: 34
Median: 71.5
Mean: 71.1
Standard Deviation: 13.9

Number receiving A's: 3 8.8%
Number receiving B's: 7 20.6
Number receiving C's: 10 29.4
Number receiving D's: 9 26.5
Number receiving F's: 5 14.7